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A History of the Teaching of Elementary Geometry

With Reference
to Present-day Problems

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PREFACE

While much has been written on the history of mathematics, comparatively little attention has been given to the history of its teaching. Günther's "Geschichte des mathematischen Unterrichts im Deutschen Mittelalter bis zum Jahre 1525" and Suter's "Die Mathematik auf den Universitäten des Mittelalters" are the only prominent works that emphasize the teaching side. Among the works that treat the general history of mathematics, those of Cantor, Hankel, Gow, and Allman have been of great assistance. Much information has also been gained from miscellaneous articles, standard works on the history of education, and early texts, the latter constituting, for the most part, the original sources for this study.

The material that has been utilized in the first three chapters is to be found chiefly in the standard histories of mathematics, but wherever possible the original sources have been consulted. Originality is claimed only for the selection and arrangement of this portion of the subject-matter and for the conclusions drawn. The material for the next three chapters has been gleaned very largely from the original sources.

The author is under great obligation to Professor David Eugene Smith of Teachers College, Columbia University, under whose direction this dissertation has been prepared, for helpful counsel and criticism, and for rendering much of this work possible through his valuable collection of early printed books.

ALVA WALKER STAMPER.

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CHAPTER I

THE TEACHING OF GEOMETRY BEFORE EUCLID

THE BEGINNING OF GEOMETRY AMONG PRIMITIVE PEOPLE

In this attempt to trace the historic development of the teaching of elementary geometry, the word teaching will be used both in its widest sense and as restricted to methods of the school room. In the more general use of the word, we shall be concerned with the manner in which man began to formulate the science, with the additions to the subject-matter in the various epochs, and with the books written to spread this knowledge. In studying the development of the teaching of geometry[1] in any epoch, four factors will be considered: 1. The contributions to the subject-matter. 2. The text-books and books read by the learned. 3. The methods of teaching geometry. 4. The place of the subject in the curriculum. The history of geometry as such will be considered only so far as is necessary for a foundation for the present study. Naturally this will form a large part of our information about the early teaching of the subject. Keeping in mind these four aspects of the matter, our subject will be treated first chronologically, bringing it down to the present time. Certain modern problems will then be considered in the light of the foregoing historic material.

Geometry as studied in the schools to-day has two values: (*a*) It is a study which has practical applications in mensuration and in the related fields of science. (*b*) It is a means of logical discipline. With respect to the general historic development of the subject-matter of geometry, we shall find: (*a*) That the practical side alone was recognized in pre-grecian geometry.

[1] Where contradictions do not arise the term geometry will generally be employed for elementary geometry.

(*b*) That geometry as a pure science (apart from its applications) was developed by the Greeks and reduced to a coherent logical system by Euclid. In accordance with these two lines of development, we shall show that two chief aims have characterized the teaching of geometry: (*a*) The practical aim, under which geometric principles have been applied in the general field of mathematics or in the related fields of science. This aim was dominant in the later Greek period, was not without influence in the Middle Ages, and has been recognized in geometry instruction in most of the countries here considered; (*b*) The disciplinary or logical aim, under which the instruction has been directly or indirectly from Euclid. Directly, where the text of Euclid has been closely adhered to; indirectly, where books based on Euclid have been followed. The opposition of the logical and practical points of view is fundamental in present-day teaching, and will occupy us constantly in this essay.

The pupil to-day is not ready for his logical geometry unless he has some practical experience on which to base his logic. It was so with man's first logical geometry. Greece based her geometry on the practical work of the Egyptians. It will therefore be necessary for us to consider this pre-grecian geometry.

Geometry arose, like all science, out of man's contact with nature. We may postulate that man's first efforts to interpret and adjust himself to nature were intuitive. We are familiar with this intuition in the habits of animals. All animals make use of what we term a geometric principle that a straight line is the shortest path between two points. Boys cross lots without first learning that one side of a triangle is less than the sum of the other two. These facts of nature are used because they are serviceable. The Indian fastened his pony to a stake, enabling him to graze in a circle. He knew that the longer the rope, the greater the area covered, but no exact relation between area and radius occurred to him. Though this early stage of intellectual development, that of intuition, does not necessarily lead up to the domain of abstract principles, still all science has its genesis in man's efforts to seek an adjustment with nature. In this instinctive stage we find man employing principles common to a higher plane of civilization. The Indian chose the cone-shaped tee-pee for economic reasons. The mound builders[1]

[1] Carr, *The Mounds of the Mississippi Valley*, p. 64.

in their ground plans employed the square and the circle, besides shapes of irregular form. Some of their mounds illustrated the truncated cone. But we cannot draw the conclusion that the mound builders had any conception of geometric principles. If they had used the principle of orientation in the building of their structures, then some real geometric notions could be traced to them. But, as far as known, such orientation was lacking.

The employment of what we call geometric design is not common to man alone. The beaver builds his dome-like structure as skillfully as the Esquimau does his. The white ants of Africa build hills twenty-five feet high, ingeniously honeycombed with galleries. While the honey bee builds his cell according to what we call geometric design, he is instead moulding nature to meet his immediate needs. In many of our so-called geometric constructions we merely copy what nature has already revealed to us. After ages of contact with nature, man was finally led to an understanding of the laws existing in and between her various forms. So nature has been the first source from which man has drawn his geometric inspirations and out of this contact with nature ideas have been developed.

Primitive man, and like him the child, does not consider *per se* the forms which nature has built around him, but observes them as necessary constituents of his well-being. He observes and thinks of the plane surface in terms of its serviceability. The same is true of the use of space forms. That there is a science in which these forms play a part is a world not yet disclosed. The same is true of the concept of number. While birds recognize a difference between a few and many grains of corn, the exact notion of how many, in all probability, never occurs to them. Even man did not need exact knowledge of number until the economic conditions of his life demanded an exact measure of discrete things. He frequently expresses his quantitative powers with respect to nature in terms of his own physical capacities, as when we speak of stone's-throw, finger-breadth, span, hand, ell, cubit, fathom, day's journey, and the like.

When the relations between space and its measure meant more to man's spiritual and physical well-being, he began to

develop a rudimentary science. When the mind came to classify, to define space relations, to summarize the products of human efforts, a second level in human experience was attained. On such a plane worked the minds of the Ancient Egyptians, the Babylonians, and the Chinese. The building of the pyramids of Egypt gave a stimulus to an architecture employing principles of proportion and the simpler facts of plane geometry. The placing of these tombs due north and south led to finding an east and west line. The effects from the overflow of the Nile demanded some knowledge of plane surveying. But the Egyptians continued on the one intellectual level. They knew enough practical geometry for their needs; they formulated rules in mensuration, thus showing an ability to classify their knowledge; but beyond this they did not go.

A third and higher level was reached by the Greeks, who, basing their work on the practical geometry of the Egyptians, developed a system of logic which culminated in the great work of Euclid. The growth of man has been in a sense like the growth of the child. First, there was the consciousness of the materials of a science. Second, the use of these materials in a practical, but not necessarily exact way. Later came the stage when the mind began to cultivate its logical powers. It was then that Greece arose with the mind of the full-grown man.

There has also been a fourth stage, that in which the theory is put to practical uses. Thus the cycle is made complete and that which arose from practical needs returns again in the form of theory to be tested and again expanded.

THE EGYPTIANS

It is well to consider, somewhat in detail, the practical geometry of the Egyptians, not only because it had a bearing on the development of the logical geometry of the Greeks but also because our story would not be complete without some statement of man's achievements in what has above been called the secondary plane of geometric development.

That the Egyptians had great mechanical skill is shown in their building arts. That a nation with so meager a knowledge of geometry built such a structure as the Great Pyramid causes astonishment. Though no great knowledge of geometry was

required for this work, we naturally query why a people capable of such mechanical achievements did not pursue further the science that was the basis for their work.

A study of the pyramids gives evidence of a knowledge both of astronomy and geometry. The Great Pyramid of Cheops, according to Charles Piazzi Smyth, gives some interesting results:[1] (*a*) The base is a square. (*b*) Its height is to twice one base edge as the diameter of a circle is to its circumference. For the truth of this statement, one can refer only to Smyth's measurements.[2] (*c*) The pyramid is orientated to within 4′ of the true north and south line. It is reasonable to suppose that the pyramid builders first located the north and south line by means of the polar star. But how did they get a perpendicular to this line? According to Cantor,[3] the philosopher Democritus (cir. 420 B.C.) is quoted as saying, "In the construction of plane figures with proof no one has yet surpassed me, not even the so-called Harpedonaptæ of Egypt." Cantor points out that the name Harpedonaptæ is made up of two Greek words meaning, when taken together, "rope-fasteners." A right angle was formed by stretching a rope around three pegs so placed that the three sides of the triangle formed were in the ratio of 3 : 4 : 5. This statement of Democritus tells us that about 420 B.C. the rope-stretchers had fame as practical geometers in Egypt. The pyramids were built during the fourth dynasty, antedating 3000 B. C. One has a right to judge that the above method of laying off perpendiculars was in use at that time, for at a little later date during the reign of Amenemhat I of the twelfth dynasty, the rope-stretchers plied their art.[4] This method of erecting a perpendicular was recognized by other

[1] Smyth, *Our Inheritance in the Great Pyramid*, pp. 11-26, 47-66.

[2] In other words here is a ratio that equals 3.14159. If we are to accept these figures and conclusions, we are to think that the Egyptians at that early time knew this close approximation to the value of π. One is inclined to think, however, that Smyth had extremely good luck in getting his results. That the Egyptians did not know the value so accurately at a later time will be shown from the records.

[3] Cantor, *Vorlesungen über Geschichte der Mathematik*, Vol. I, p. 62. Hereafter referred to as Cantor.

See Gow, *History of Greek Mathematics*, p. 129. Hereafter referred to as Gow.

[4] Dümichen, *Denderatempel*, p. 33.

nations. The Hindus made use of it, and the Chinese are credited with knowing that a triangle is right angled when the sides are in the ratio 3 : 4 : 5.[1] It is hardly necessary to add that the above method of erecting perpendiculars is in common use to-day.

Piazzi Smyth gives us further information regarding the Great Pyramid at Cheops: (*d*) The angle of the entrance passage on the north is a little over 26°. (*e*) The angle of the northern air passage is 33° 42′.[2] According to Colonel Howard-Vyse's statement[3], in 2400 B.C. the lower culmination of the polar star was 26°. So Piazzi Smyth draws the conclusion that these passage ways pointed to the lower and upper culminations of the polar star in the year 2400 B.C. The average of the above figures gives an approximation of 30° for the latitude of the Great Pyramid. The latitude as obtained by the French[4] in 1799 was found to be 29° 59′ 6″. We thus see that the Egyptians had knowledge of mathematical astronomy as well as of geometry in the building of this pyramid.

The Egyptians were also stimulated to use a form of geometry due to æsthetic influences. In their mural decorations during the period of the fifth dynasty immediately following the building of the Gizeh pyramids, we find evidences of geometric designs embodying principles of symmetry. This is found in particular in the square and its diagonals, the rhombus, and the isosceles trapezoid. One figure shown by Professor Cantor[5] represents two squares one over-lapping the other, so placed as to give the effect of an eight-pointed star. There is also represented the division of the circle into 4, 8, 6, 12, parts by the requisite number of diameters.[6] All of this was accomplished before Greece became interested in geometry. The probable effect of such work on Greece is pointed out by Gow, who says: "To a Greek, therefore, who had once acquired a taste for geometry, a visit to Egypt or Babylon would reveal a hundred

[1] Gow, p. 130.

[2] These angles are referred to the horizon.

[3] Smyth, *op. cit.*, p. 57.

[4] *Ibid*, p. 65.

[5] Cantor, I, pp. 66-67.

[6] The Babylonians divided the circumference into 360° and also made use of various geometric designs in their mural decorations. Cantor, I, p. 98.

geometrical constructions which, on inspection, suggested new theorems and invited scientific inquiry." [1]

The Egyptians were also stimulated to a use of geometry for economic reasons. Herodotus[2] tells us that Rameses II (cir. 1400 B.C.) divided the land of Egypt into equal squares so as to provide a more convenient means of taxation. But on account of the frequent overflows of the Nile, parts of these subdivisions were frequently swept away, and the king appointed surveyors to levy the proper tax in proportion to the part of the land remaining. There were certain benefits to be derived from the overflow of the Nile, and these again necessitated government supervision. Wilkinson says:[3] "Besides the mere measurement of superficial areas, it was of paramount importance to agriculture . . . to distribute the benefits of the inundation in due proportion to each individual, that the lands which were low might not enjoy the exclusive advantages of the fertilizing water, by constantly draining it from those of a higher level. For this purpose, the necessity of ascertaining the various elevations of the country, and of constructing accurately levelled canals and dykes, obviously occurred to them. . . . These dykes were succeeded or accompanied by the invention of sluices, and all the mechanism appertaining to them; the regulation of the supply of water admitted into plains of various levels, the report of the exact quantity of land irrigated, the depth of the water admitted into plains of various levels, and the time it continued upon the surface, which determined the proportionate payment of the taxes, required much scientific skill." We thus see the development of a rudimentary surveying and with it the necessary practical geometry.

That the Egyptians considered their geometric learning to be worthy of preservation is shown from the contents of an old papyrus,[4] which is now in the Rhind collection in the British Museum. This papyrus was written about 1700 B.C.

[1] Gow, pp. 132-133.

[2] Herodotus II, 109.

[3] Wilkinson, *The Ancient Egyptians*, Vol. IV, pp. 7-8.

[4] This was translated in 1877. See Eisenlohr, *Ein Mathematisches Handbuch der alten Aegypter*, pp. 125-150. There is a supposition that this is a copy of a much older work. For a copy of the original, see *Facsimile of the Rhind Mathematical Papyrus in the British Museum*. This contains a list of the principal works that describe the papyrus.

by the scribe Ahmes. Here are given rules for finding the areas of some of the plane figures, those for the areas of isosceles triangles and isosceles trapezoids being incorrect.[1] Figures are given for these as well as for squares, rectangles, and circles. The diagrams are not lettered, but the lengths of various lines are indicated by means of the hieratic number symbols, which are copied on the figures just as is done to-day. As this is the oldest mathematical writing that has been deciphered, we have here the earliest known drawing of geometric figures employed for educative purposes.[2] These incorrect rules in mensuration were used much later in some of the Egyptian temple inscriptions. The Temple of Horus at Edfu in Upper Egypt gives evidence of this fact. These inscriptions (cir. 100 B.C.) describe the lands "which formed the endowment of the priestly college attached to the temple."[3] The incorrect formula for the isosceles trapezoid was also applied to any trapezoid. The formula for the general quadrilateral was $\frac{a+b}{2} \cdot \frac{c+d}{2}$, where the four sides a, b, c, d are in general unequal.

Returning to the Ahmes papyrus, we find that there is given a formula for determining the area of a circle. Ahmes takes a circle whose diameter is 9 units and writes the area 64, using as a formula $(D - \frac{1}{9} D)^2 =$ Area. This is equivalent to taking $\pi = 3.1605$, which shows a remarkably close approximation to the correct value. The papyrus also gives calculations for finding the contents of certain barns the shapes of which it does not accurately describe, and adds some examples on pyramids

[1] Ahmes considers an isosceles triangle whose base is 4 units in length and whose leg is 10 units. The area is given as 20, showing that his formula is equivalent to multiplying the length of one of the equal legs by one-half the length of the base. He takes an isosceles trapezoid whose parallel bases are respectively 6 and 4 units each long. One of the equal legs is 20 units. The area is given equivalent to the form $\frac{4+6}{2} \times 20$.

[2] There is in the British Museum a MS. written on leather, which, it is claimed, pertains to early Egyptian mathematics. It was sufficiently examined to convey this impression, but on account of the leather tending to crumble, it cannot be examined further.

[3] Gow, p. 131; Hankel, *Zur Geschichte der Mathematik in Altertum und Mittelalter*, pp. 86, 87. Hereafter referred to as Hankel.

which employ a rudimentary trigonometry in which the hypotenuse and base of right triangles are given to find their ratio (seqt).[1] This ratio determined the cosine of an angle, which, for all the pyramids, gives practically the same slant of the lateral faces. The term seqt was also used to denote the ratio which determines the tangent of an angle.

So much for the development of geometry by the Egyptians. They allowed the Greeks to come and learn, but just how this knowledge was communicated is hard to say. As the priests constituted the learned class, undoubtedly it came from them. Perhaps, also, the Greeks got some inspiration, as Gow relates, by observing the geometric constructions on the walls of the temples. Besides this, the architecture of the various temples would have given some instruction.

One could well ask why the Egyptians did not develop a logical geometry. Gow very properly states: "It will readily be supposed that the Egyptians, who had so early invented so many rules of practical geometry, could not fail in process of time to make many more discoveries of the same kind, and thus be led to geometrical science. But it appears that in Egypt, land-surveying, along with writing, medicine and other useful arts, was in the monopoly of the priestly caste; that the priests were the slaves of tradition, and that, in their obstinate conservatism, they were afraid to alter the rules or extend the knowledge of their craft. Of their medicine, Diodorus (I. 82) expressly relates that, even in his day, the Egyptian doctors used only the recipes contained in the ancient sacred books, lest they should be accused of manslaughter in case the patient died. Geometry seems to have been treated with similar timidity."[2]

In brief, the Egyptians knew how to calculate the areas of some of the simple rectilineal figures, using some rules, however, that were erroneous. Also, they found the capacity of barns by methods not clearly defined. In some of their problems on pyramids the idea of ratio was involved. Finally, they employed some principles of symmetry in their mural decorations. On the whole, the Egyptians developed a practical geometry of areas.

[1] Cantor, I, pp. 59-60; Gow, pp. 128-129. Cantor and Eisenlohr have worked out these interpretations.

[2] Gow, p. 130.

As for the methods of instruction employed by the Egyptians, nothing is definitely known. With the exception of the manuscript of Ahmes, the undeciphered manuscript mentioned above, and some temple inscriptions, we have no record of Egyptian geometry from native sources.

THE GREEKS BEFORE EUCLID

The Development of the Subject-Matter of Elementary Geometry

The Greeks, who were the first to study geometry from a logical viewpoint, brought the subject into a coherent system during a period of 300 years. This development began when Thales in the capacity of a merchant visited Egypt[1] and there found the materials upon which to base his science, and culminated when Euclid (cir. 300 B.C.) wrote his "Elements." On his return to Asia Minor, Thales founded the Ionian School of mathematics and philosophy and was there visited by Pythagoras, who, after traveling in Egypt and perhaps Babylonia,[2] founded his school at Croton in Southern Italy. When the Pythagorean school declined, after the death of its founder, the seat of learning was changed to Athens, which was then in the height of its power.

Geometry was studied arduously there by the Sophists, and, contemporaneous with them, Plato and his pupils contributed to the progress of the science. After the time of Aristotle we again see the study of geometry thrive on Egyptian soil at the newly founded city of Alexandria. In this Grecian city the logic of geometry was to be rounded out through the work of Euclid, whose text has influenced so largely the teaching of geometry even up to the present day. Let us now look into the additions to the subject-matter of geometry during this period of 300 years before Euclid, and also see what contributions were made from the standpoint of method.

Thales and his school have been credited with having added to geometry these five theorems:[3] (1) A circle is bisected by its diameter; (2) the angles at the base of an isosceles triangle are

[1] This is mentioned in the Eudemian Summary. See below p. 14.

[2] Cantor, I, pp. 138-141; Gow, pp. 66, 148.

[3] Cantor, I, pp. 124-136; Gow, pp. 140-145.

equal; (3) if two straight lines intersect the vertical angles are equal; (4) an angle inscribed in a semicircle is a right angle; (5) a triangle is determined if its base and base angles are known. That Thales did not prove all the propositions attributed to him is shown by a statement from the Eudemian Summary,[1] that Euclid first thought the third worthy of proof.[2]

Although Thales was interested in the development of logical geometry, he was primarily an astronomer, and no doubt was impelled to further study of geometry by recognizing the relation between theory and practice. His practical turn of mind is referred to by Eudemus, who credited him with inventing a way of finding the distance of a ship at sea.[3] The principle involved is associated with his fifth proposition mentioned above. Thales is also credited with finding the heights of pyramids by means of shadows.[4]

Œnopides of Chios (cir. 450 B.C.), who seems to have been associated with the Ionic school, contributed to the development of geometry. According to Proclus,[5] he solved the two problems: "From a point without a straight line of unlimited length to draw a straight line perpendicular to that line," and "At a given point in a given straight line to make an angle equal to a given angle." Concerning the first of these problems, Proclus says that Œnopides first invented this problem, thinking it useful for astronomy. This is interesting, for it shows that the early Greeks did not entirely ignore practical geometry, and in particular we see the stimulating influence of science.

The work of Thales and the Ionian school was both practical and theoretical. While the Egyptians were concerned with areas in their practical work, Thales in his logical work developed theorems concerned with lines, which required a high degree of abstraction. To Thales, then, we can attribute the beginning of the geometry of lines and with it the deductive method of reasoning applied to geometry.

The next advance of any importance was made by Pythagoras, who seems to have been more directly influenced by the Egyp-

[1] See below, p. 14.

[2] Proclus, ed. Friedlein, p. 299.

[3] *Ibid*, p 352.

[4] Pliny, *Natural History*, trans. Bostock and Riley, xxxvi, 17

[5] Proclus, ed. Friedlein, pp. 283, 333.

tians than was Thales. This is seen by the attention he gave to the geometry of areas and volumes, and to arithmetic. The school of Pythagoras was undoubtedly familiar with many of the propositions in the first two books of Euclid and with parts of the fifth and sixth books;[1] that is, it was familiar with the ordinary theorems in plane geometry concerning equality of lines and of angles and with many of the theorems on equivalent and congruent areas. The geometry of the circle was not developed by the Pythagoreans, but was studied later at the Athenean school. It is said that the Pythagoreans knew that the angle sum of a triangle equals two right angles. Hence they held a conception of parallel lines if we are to suppose that any proof of the above theorem was accomplished. The theorem of the three squares (Euclid, I, 47), that in any right triangle the square on the hypotenuse equals the sum of the squares on the other two sides, is known as the Pythagorean theorem. The Egyptians knew the truth of this theorem where the sides were in the ratio of 3 : 4 : 5, but Pythagoras was the first to see the truth of this relation in any right triangle.

Pythagoras is also credited with the discovery of the geometric irrational and of the three kinds of proportion, arithmetical geometric, and harmonic. The Pythagoreans were much interested in the study of the regular solids and are credited with their constructions. This being true, they were certainly familiar with the construction of the regular plane polygons of 3, 4, 5, sides. The construction of the regular polygon of five sides depends upon the division of a line in extreme and mean ratio. Allman[2] contends that Pythagoras was familiar with this, but Gow[3] quotes from the Eudemian Summary, which attributes the discovery of this problem, known as the Golden Section, to the school of Plato. Gow's conclusion admits of less speculation and perhaps is nearer the truth.

We must note that in all that is known regarding the contributions of Thales and Pythagoras to the development of geometry there is more or less speculation as to the nature of the material, but in particular there is no definite statement re-

[1] Gow, p. 153.

[2] Allman, *Greek Geometry from Thales to Euclid*, p. 40. Hereafter referred to as Allman

[3] *Op. cit.*, p. 153.

garding the methods of proof. We are, perhaps, safe in judging that a large part of the work of these early schools was in finding out geometric truths, and that the sequence of proved theorems was not then held in any hard and fast line.

We have no record that the school of Pythagoras was concerned with the practical, and we may conclude with Dr. Allman[1] that the Pythagoreans were the first to sever geometry from the needs of practical life and to treat it as a liberal science.

Thus far we have the growth of geometry—first, the practical stage as with the Egyptians; second, the beginning of the logical stage in the school of Thales, where practical applications were employed and the foundations of deductive geometry laid; and thirdly, under the Pythagoreans, we have the subject treated as a liberal science. But we have no proof that the subject-matter was yet organized into any fixed sequence.

When the Pythagorean school at Croton in Southern Italy was disbanded for political reasons, its influence had already grown and other schools had been founded on the shores of the Mediterranean. About this time, fresh from her glories of the Persian wars, Athens, exceedingly wealthy, attracted people of all nations. Among these were teachers who were willing to work for hire. Such were the Sophists. To them and the school of Plato, we are principally indebted for the great mass of subject-matter which was finally organized into a text by Euclid of Alexandria. We recall that the Pythagoreans developed the geometry of areas but neglected the geometry of the circle. This study was taken up by the Athenian Greeks and many theorems were discovered in their futile attempts to solve the so-called Three Problems of Antiquity: the trisecting of any angle, the duplication of the cube, and the quadrature of the circle.[2]

Something of the nature of the contributions to the subject-matter of geometry during this period can be seen from the titles of some of the works. Euclid is universally credited with being the first to write a complete text on geometry, but he was not the first to write on particular portions of it. Although the school of Thales is not generally credited with adding a great

[1] *Op. cit.*, p. 47.

[2] For a scientific treatment of these famous problems, see Klein, *Famous Problems of Elementary Geometry*, trans. Beman and Smith.

deal to the subject-matter of geometry, to two of its members have been attributed geometric writings. According to Suidas,[1] Anaximander wrote a work with the title "A Collection of Figures Illustrative of Geometry," and Plutarch[2] (De exilio c. 17) states that Anaxagoras of Clazomenæ wrote a treatise on the quadrature of the circle. According to Vitruvius[2] (vii. Praef.), Anaxagoras also wrote a work on perspective. The Pythagoreans published nothing of their work on geometry, although later Philolaus, who lived in the time of Plato, published an account of their philosophy.[3] What we know of the writings on geometry during its development at Athens is to be found in the Eudemian Summary to which reference has already been made:[4] " . . . Hippocrates of Chios, next, who discovered the quadrature of the lune, and Theodorus of Cyrene became distinguished geometers; indeed Hippocrates was the first who is recorded to have written 'Elements.' Plato, who followed him, caused mathematics in general, and geometry in particular, to make great advances, by reason of his well-known zeal for the study, for he filled his writings with mathematical discourses, and on every occasion exhibited the remarkable connexion between mathematics and philosophy. To this time belong also Leodamas the Thasian and Archytas of Tarentum and Theætetus of Athens, by whom mathematical inquiries were greatly extended, and improved into a more scientific system. Younger than Leodamas were Neocleides and his pupil Leon, who added much to the work of their predecessors: for Leon wrote an 'Elements' more carefully designed, both in the number and the utility of its proofs, and he invented, also, a *diorismus* (or test for determining) when the proposed problem is possible and when impossible. Eudoxus of Cnidus, a little later than Leon, and a student of the Platonic school, first increased the number of general theorems, added to the three proportions three more, and raised to a considerable quantity the learning, begun by Plato, on the subject of the (golden) section,[5] to which he applied the analytical method. Amyclas of Heraclea, one of Plato's

[1] Gow, p. 145.

[2] *Ibid.*, p. 146.

[3] *Ibid.*, p. 149.

[4] *Ibid.*, pp. 135-137 (ref. *Proclus*, ed. Friedlein)

[5] The cutting of a line in extreme and mean ratio.

companions, and Menæchmus, a pupil of Eudoxus, and a contemporary of Plato, and also Deinostratus, the brother of Menæchmus, made the whole of geometry yet more perfect. Theudius of Magnesia made himself distinguished as well in other branches of philosophy as also in mathematics; composed a very good book of 'Elements,' and made more general propositions which were confined to particular cases. Cyzicenus of Athens also about the same time became famous in other branches of mathematics, but especially in geometry. All these consorted together in the Academy and conducted their investigations in common. Hermotimus of Colophon pursued further the lines opened up by Eudoxus and Theætetus, and discovered many propositions of the 'Elements' and composed some on *Loci*. Philippus of Mende, a pupil of Plato and incited by him to mathematics, carried on his inquiries according to Plato's suggestions and proposed to himself such problems as, he thought, bore upon the Platonic philosophy."

Besides these, others have been credited with writings on geometry. Diogenes Laertius relates that Democritus wrote on geometry, on numbers and perspective, and also two books on incommensurable lines and (?) solids.[1] Another work ascribed to him bears the incomprehensible title, "The Difference of the Gnomon or the Contact of the Circle and the Sphere." Suidas[2] credits Theætetus with having written on the five regular solids. Theophrastus,[3] a pupil of Aristotle, according to Diogenes Laeritus, wrote a history of geometry in four books, together with six books of astronomy and one of arithmetic. According to Hypsicles,[4] a book on "The Comparison of the Five Regular Solids," was written by Aristæus. This contained the theorem, "The same circle circumscribes the pentagon of the dodecahedron and the triangle of the icosahedron, these solids being inscribed in the same sphere."

As has already been mentioned, during the pre-euclidean period the subject-matter of elementary plane geometry was practically completed. In the time of the Pythagorean school, the five regular solids were studied, but stereometry as a science was not

[1] Gow, p. 159 (ref. Diog. L., ix, 7, 47)

[2] *Ibid.*, p. 183.

[3] *Ibid.*, p. 134.

[4] Allman, p. 198.

yet established. The school of Plato also studied the regular solids.[1] A decided advance was made in this study when Eudoxus[2] proved the theorem, "A triangular pyramid is one-third of a prism on the same base and between the same parallels." [3] This established a relation between magnitudes of apparently unlike properties and hence tended to systematize the logic of solids. And so Eudoxus may be credited with having founded the science of solid geometry.

From our account thus far, we can see that the development of elementary geometry was not parallel with the sequence exhibited in the "Elements" of Euclid. In other words, this sequence was not according to the historic development of the subject. This will be considered later, when treating the sequence of Euclid; but may be illustrated briefly at this point. The subject-matter of the third book of Euclid, which treats of circles, was developed in the Athenian period. The theory of proportion which follows later in the "Elements" was systematized earlier by the Pythagorean school. This school also studied the properties of the five regular solids before the great amount of material of plane geometry had yet been worked over, much less systematized. This lack of agreement between the historic development of geometry and the common logical sequence will be referred to in the last chapter of this essay.

Educational Features of the Greek Geometry

It is not to be expected in the early development of any science that any but mature minds should engage in its study. So it was in the study of geometry by the Greeks. We know nothing of the conduct of the school of Thales at Miletus, but undoubtedly the word "school" is to be used only in the sense of a select body of men, probably few in number, working together

[1] They have since been called the Platonic Bodies.

[2] Allman, p. 88.

[3] Archimedes states that Democritus was the first to state the above as a formula without proof. See Heiberg's German translation of the Greek text of a newly discovered MS. by Archimedes, published in *Bibliotheca Mathematica*, vol. 7. For a brief account of the same in English see an article by Professor Charles S. Slichter in *Bulletin of the American Mathematical Society*, vol. XIV, No. 8.

in the same field of investigation, recognizing as their leader the one whose wisdom was the greatest. It was the same in the school of Pythagoras, in Southern Italy. We learn[1] that the master lectured there on philosophy and mathematics, and that his listeners were of two kinds. In the first class the lectures were of a more general nature, and women were allowed to attend. In the second class women were debarred. We are to judge that only the select who had some skill as mathematicians and philosophers were allowed within the inner circle. Here the various contributions to the new science were made and the members took upon themselves a vow not to reveal these discoveries to the world.

The Sophists are credited with having introduced higher education into Athens. There was a demand for the study of philosophy and mathematics, and so the Sophists came. Their schools were on the street corners or within the gymnasia. Their method was the Socratic, that of question and answer. So Socrates, who was by no means a Sophist, furnished a method of study that endures even to-day. Under such a developmental method the elements of geometry stood more ready to be appreciated by minds not yet matured. One has good reason then to believe that in the time of Plato and the Sophists, the study of geometry was becoming more common.

That the Greek mind was more interested in the chain of reasoning than in the subject-matter itself is illustrated in some of the dialogues of Plato.[2]

Under the old Greek education, the youth from sixteen to eighteen continued his physical and social training. Under the new education more attention was given to the training of the intellect.[3] We learn from the 'Republic" something of the place of geometry in the curriculum proposed by Plato.[4] The pupil from the age of seven to about seventeen was to study music and gymnastics. Under music was included reading, writing, arithmetic, and geometry. In this early period there was to

[1] Ball, *A Short Account of the History of Mathematics*, pp. 19-22.

[2] See the dialogue on incommensurable lines between Socrates and one of Meno's slaves in Plato's *Meno*, 81-85, trans. Jowett.

[3] Monroe, *A Text-book in the History of Education*, p. 115

[4] *Republic*, II–IV; VII, 537-540, trans. Jowett. See Bosanquet, *The Education of the Young in the Republic of Plato*, pp. 1-14.

be no compulsion, it being sufficient that the student enter into the work as dictated by his interest and his aptitude. From seventeen to twenty the youth received the military gymnastic training of the ephebe. From twenty to thirty, the study of arithmetic, geometry, music, and astronomy was undertaken in a thoroughly systematic manner for those having special fitness and inclination. At the age of thirty-five, after five years' study of dialectics, the trained man returned to social and political life and was to be there a directing force. In the "Laws," written in the old age of Plato, there are the same general recommendations, only the later years of the man are to be devoted to mathematical and astrological studies.

The four-fold division of the mathematical sciences was instituted by the early Greeks. We have observed in the recommendations of Plato the study of arithmetic, geometry, music, and astronomy. Later in Rome, Cassiodorus embodied them in the quadrivium, which constituted the advanced course in the medieval monastic schools. But the Pythagoreans were the first to assign this four-fold division. From Proclus[1] we learn that "the Pythagoreans made a four-fold division of mathematical science, attributing one of its parts to the *how many*, and the other to the *how much;* and they assigned to each of these parts a two-fold division. For they said that discrete quantity, or the *how many*, either subsists by itself, or must be considered with relation to some other; but that continued quantity, or the *how much*, is either stable or in motion. Hence they affirmed that arithmetic contemplates that discrete quantity which subsists by itself, but music that which is related to another; and that geometry considers continued quantity so far as it is immovable; but astronomy contemplates quantity so far as it is of a self-motive nature." Thus we see the origin of the plan of keeping motion out of the domain of geometry. It was essentially a study of forms in fixed positions. The idea of rotation, which is employed to-day in the study of elementary geometry, was therefore not permissible. The Greeks throughout were consistent with these ideas. The Pythagoreans founded the theory of proportion, giving a method applicable in the fields of both arithmetic and geometry, but, notwithstanding this close relationship, geometry

[1] Allman, p. 23 (ref. *Proclus*, ed. Friedlein, p. 45).

with the Greeks was developed within itself. Algebra was not yet invented, and a great deal of mathematical work, since simplified by the methods of algebra, was laboriously carried on by geometry alone. The idea of a duality, of a one-to-one correspondence between algebra and geometry, could not be utilized, and not until the time of Descartes (1637) was this principle fully recognized. Aristotle was more practical than some of his predecessors, but he showed this same tendency to confine each branch of mathematics to its own domain. He says,[1] "We cannot prove anything by starting from a different genus, e.g., nothing geometrical by means of arithmetic. . . . Where the subjects are so different as they are in arithmetic and geometry we cannot apply the arithmetical sort of proof to that which belongs to quantities in general, unless these quantities are numbers, which can only happen in certain cases." The theory of geometry was thus isolated by the Greeks from other branches of mathematics, and mathematical development was thereby retarded many centuries.

The restriction that problems of construction in elementary geometry should be limited to the use of the rule and compasses as instruments of construction dates from the Grecian period. Such a restriction barred out all so-called mechanical constructions, and as a result it was impossible to get a solution of the three famous problems, trisecting any angle, finding a square equal in area to a given circle, and duplicating a given cube. These were capable of solution by means of conic sections and certain special curves, but not by using circles and straight lines. Gow is perhaps right in attributing this restriction to the influence of Plato.[2] Eutocius (Torelli ed., p. 135) relates that Plato invented a mechanical device for inserting two mean proportionals between two given straight lines, Hippocrates having reduced the duplication problem to this one. But Plutarch in two of his writings relates "that Plato blamed Archytas and Eudoxus and Menæchmus for using such instruments for the purpose of solving the duplication-problem, and said that the good of geometry was spoilt and destroyed thereby,"[3] and that owing to this remonstrance of Plato, "mechanics were separated from geom-

[1] Allman, p. 146 (ref. *Anal. post.* I, vii, p. 75a, ed. Bek.).

[2] Gow, p. 181.

[3] *Ibid.* (ref. *Quaest. Conv.* viii, 9, 2, c. 1).

etry and became a branch of the military art."[1] Here are two opposing statements, but Plato's general position as to the aim of education would lead one to the conclusion that the statements attributed to him are the correct ones, and that at least as early as his time the above restriction was imposed in the use of instruments in elementary geometry. Thus we see the expulsion of mechanics from geometry. But mechanics was studied nevertheless, and the science was developed under Aristotle and later made great progress at Alexandria. Archytas is mentioned as the first to place mechanics on a scientific basis and "to apply mechanical motion to the solution of a geometric problem, while trying to find by means of the section of a semi-cylinder two mean proportionals with a view to the duplication of the cube."[2]

During this period there appears to have been a neglect of solid geometry. It has already been mentioned that Hippocrates reduced the problem of the duplication of the cube to that of finding two mean proportionals between two given straight lines. He thus changed the problem from one of solid geometry to one of plane. This indicates a tendency of the times, or else Plato[3] would not have complained that stereometry went entirely out of fashion. Furthermore, we know that by the time of Euclid solid geometry was not developed in the sense that characterized plane geometry.

Although they made no use of them in theoretical geometry, we may mention that the Greeks were familiar with the square, the level, and the gnomen or carpenter's square. The inventions of the square and level are attributed by Pliny (Nat. Hist. VII, 57) to Theodorus of Samos, who lived contemporaneous with Thales.[4] According to Allman, they were known long before to the Egyptians: "So that to Theodorus is due at most the honour of having introduced them into Greece." Anaximander, of Miletus, was the first to introduce the gnomen and the sun-dial into Greece.[5] These came according to Herodotus, from the Babylonians[6]

[1] Plutarch *Marcellus*, trans Langhorne, p. 106.

[2] Allman, p. 110, quoting from Diog. Laert.; Plutarch, *Marcellus*, trans. Langhorne, p. 106.

[3] *Republic*, VII, 528, trans. Jowett.

[4] Allman, p. 15. [5] Gow, p. 145. [6] Herodotus, II, 109.

The lettering of geometric figures began with the Greeks. Cantor points out that the letters spelling the word "health" were placed at the vertices of the pentagram, the Pythagorean emblem.[1] Hippocrates, later, in attempting the quadrature of the lune, used quaint descriptions in describing the lettering of his figures. Thus, he wrote, "the line on which AB is marked" and "the point on which K stands."[2] Aristotle still later used a letter symbolism in his Physics (VII, 5, pp. 249-250 of the Berlin ed.). He says, "If A be the mover, B the moved thing, Γ the distance, and Δ the time of the motion, then A will move $\frac{B}{2}$ twice the distance Γ in the time Δ or the whole distance Γ in half the time Δ." The value of this symbolism was fully appreciated by Aristotle when he said that much time and trouble is saved by a general symbolism.[3]

Recognizing the Greek love of oratory and in particular the propensity of the Sophists for verbal disputations, we can judge that geometry was taught orally, and only that was committed to writing which was no longer a subject for argument. It is probable that a board strewn with sand was in common use for the drawing of figures, for this was a common method among the orientals in doing their calculating, and the Greeks certainly made use of this convenient mode.[4]

It will be pointed out later that the tendency to hold to the special characterized the practical geometries of Latin Europe even up to the middle of the seventeenth century. The period was one of retrogression in this respect when compared with the time of the early Greeks. But we shall also see that Euclid was not entirely free from this tendency. This being the case, we should expect to find in the pre-Euclidean geometry like tendencies. "Eutocius, at the beginning of his commentary on the Conics of Apollonius (p. 9, Hallev's edn.), quotes from Geminus.

[1] Cantor, I, p. 195.

[2] Gow, p. 169.

[3] *Ibid.*, p. 105n.

[4] There is a tradition, that as Archimedes was contemplating some geometric figures drawn in the sand on the floor, a soldier was admitted and ordered Archimedes to follow him to Marcellus. Archimedes, refusing to do it until he had finished his problem, the soldier in a passion drew his sword and killed him. Plutarch, *Marcellus*, trans. Langhorne, p. 114.

an excellent mathematician of the first century B.C., the following remarks: 'The ancients, defining a cone as the revolution of a right-angled triangle about one of the sides containing the right angle, naturally supposed also that all conics are *right* and there is only one kind of section in each—in the *right-angled* cone the section which we now call a parabola, in the *obtuse-angled* a hyperbola, and in the *acute-angled* an ellipse. You will find the sections so named among the ancients. Hence just as they considered the theorem of the two right angles for each kind of triangle, the equilateral first, then the isosceles, and lastly the scalene, whereas the later writers stated the theorem in a general form as follows, 'In every triangle the three interior angles are equal to two right angles,' so also with the conic sections, they regarded the so-called 'section of a right-angled cone' in the right-angled cone only, supposed to be cut by a plane perpendicular to one side of the cone: and similarly the sections of the obtuse-angled and acute-angled cones they exhibited only in such cones respectively, applying to all cones cutting planes perpendicular to one side of the cone. . . . But afterwards Apollonius of Perga discovered the general theorem that in every cone, whether right or scalene, all the sections may be obtained according to the different directions in which the cutting plane meets the cone.' "[1] While the early treatment of conic sections shows the difficulties in reaching the conception of the general, it is the same treatment of the theorems in elementary geometry that is of interest to us here. We thus see the early difficulties of the race; the tendency to pass from the special to the general is not only characteristic of the individual in his development, but of the race also.

As to the conciseness of geometric proofs, it is probable that compared with our practice to-day there was little reference to previous theorems, and hence a certain amount of tediousness was avoided. Even Euclid, who systematized the existing geometry, referred to previous propositions in only an indirect way. But to the learners of geometry in these early times, the method must have seemed far from being concise, where necessary steps in the proofs were insisted upon. Alexander the Great, who had Menæchmus for a teacher, complained of the length of the proofs. When he asked his teacher if the instruction could not

[1] Gow, p. 137.

be made somewhat shorter, Menæchmus replied, "O King, in the material world there are roads for common people and roads for kings, but there is only one road to geometry, and that is for all."[1] A similar story is told of Euclid in reply to the question of King Ptolemy.

The geometers before Euclid have given us plans of attack that are standard to-day. We shall refer in particular to the notion of locus, the method of exhaustion, reductio ad absurdum, and analysis.

Allman[2] contends that the conception of geometric locus is due to Thales. He bases his conclusion in part on the fact that Thales knew that any angle inscribed in a semi-circle is a right angle. Dr. Allman may be right, but his conclusions do not seem warranted. It seems probable though, that the notion of geometric locus was understood before the time of Archytas (cir. 400 B.C.), for Archytas not only employed this idea, but used the intersection of loci for the determination of a point.[3] We learn also from the Eudemian Summary that Aristæus (cir. 320 B.C.) wrote on solid loci, and later Hermotimus of Colophon composed some propositions on loci. About this time curves of all kinds were called *running loci*, the straight line and the circle were called *plane loci*, and the conic sections *solid loci*.[4] At least by the time of Plato, the conception of geometric locus was fully appreciated.[5]

Eudoxus is generally credited with the perfection of the method of exhaustion. Before him Antiphon and Bryson had employed the *process* of exhaustion in seeking the quadrature of the circle. This was the process of exhausting the area of a circle by means of inscribed and circumscribed polygons. Antiphon used only the series of inscribed polygons, while Bryson used both the inscribed and circumscribed. The latter, the more rigid, was adopted by Euclid, and we find it in use to-day in many of our texts. But the so-called method of exhaustion

[1] Bretschneider, *Die Geometrie und die Geometer vor Euklides*, pp. 162-163

[2] *Op. cit.*, p. 13.

[3] See Allman, pp. 111-114. Archytas effects the duplication of the cube by the intersections of a cylinder, cone, and hemisphere.

[4] Gow, p. 187n.

[5] See Montucla, *Histoire de mathématiques*, Tome I, p. 183; Chasles, *Aperçu historique des methodes en géometrié*, p. 5.

was established later by Eudoxus. It embodies the two propositions mentioned below.[1] In establishing the method of exhaustion, Eudoxus, as we can see in the propositions cited, made use of the *reductio ad absurdum*, which is a most powerful instrument of attack in mathematical theory. In this connection we should mention the use of geometric *reduction*. This idea is so common to-day that we do not dignify it with a definition. Proclus,[2] discussing the recasting of the duplication problem by Hippocrates, credits him with the invention of this method, which he defines as a transition from one problem or theorem to another, which, being solved or proved, the thing proposed necessarily follows. This method then was employed

[1] I. If from A more than its half be taken, and from the remainder more than its half, and so on, the remainder will at last become less than B, where B is any magnitude named at the outset (and of the same kind as A), however small. II. Let there be two magnitudes, P and Q, both of the same kind; and let a succession of other magnitudes, called X_1, X_2, X_3, . . . be nearer and nearer to P, so that any one, X_n, shall differ from P less than half as much as its predecessor differed. Let Y_1, Y_2, Y_3, ... be a succession of quantities similarly related to Q; and let the ratios X_1 to Y_1, X_2 to Y_2, and so on, be all the same with each other, and the same with that of A to B. Then it must be that P is to Q as A to B. (It is obvious, from the conditions, that if X_1 be greater than P, Y_1 is greater than Q, etc.) Suppose X_1, X_2, etc., less than P, and therefore Y_1, Y_2, etc., less than Q. Then if A is not to B as P to Q, A is to B as P to some other quantity S greater or less than Q; say, less than Q. Then (by hyp. and I) we can find some one of the series Y_1, Y_3,, . . (say Y_n) which is nearer to Q than S is to Q and which is therefore greater than S. Then since X_n is to Y_n as A to B, or as P to S, we have X_n is to Y_n as P to S, or X_n to P as Y_n to S; from which, since X_n is less than P, Y_n is less than S. But Y_n is also greater than S, which is absurd; therefore, A is not to B as P to less than Q. Neither is A to B as P to more than Q (which call S), for in that case S is to P as B to A; let S be to P as Q to T, then S is to Q as P to T; from which, S being greater than Q, P is greater than T. But B is to A as S to P, that is, as Q to less than P, which is proved to be impossible by the reasoning of the last case. Consequently, A is not to B as P to more than Q, or to less than Q; that is, A is to B as P to Q, which was to be shown. Let P and Q be two circles, A and B the squares on their diameters, X_1 and Y_1 inscribed squares, X_2 and Y_2 inscribed regular octagons, X_3 and Y_3 inscribed regular figures of sixteen sides, etc., the preceding process gives the proof that circles are to one another as the squares of their diameters. See De Morgan's article, *Geometry of the Greeks*, in the Penny Encyclopedia, and Gow, pp. 171, 172.

[2] Proclus, ed. Friedlein, p. 212.

in the discovery of new theorems or problems. It sets up a chain of steps by which certain conclusions are reached. This we see prepares the way for the reductio ad absurdum which in turn embodies the idea of *analysis*.

As we know, the method of analysis is employed by supposing for the time being that the desired theorem is true or the sought problem is effected so as to find the necessary underlying conditions. In reductio ad absurdum the contrary theorem is proved to be not true by analysis. We must also then credit Eudoxus with employing this method, for the method of exhaustion involves the reductio ad absurdum, and it in turn employs the method of analysis. "Both Proclus (ed. Friedlein, p. 211), and Diogenes Laertius (III, 24), state that Plato invented the method of proof by analysis."[1] Plato and Eudoxus were contemporaries. We have positive evidence of Eudoxus employing this method, as shown above, but Proclus and Diogenes Laertius give the credit to Plato. It is most probable that Plato systematized the method and gave it a definite form.

The *discussion* of geometric problems owes its origin to the Greeks. We are accustomed to-day to attack a problem by analysis, then, after the necessary conditions are found, the construction is made, the deductive proof is given, and then is added the discussion of the conditions under which the problem is or is not solvable. The Greeks called this the *diorismus*. The Eudemian Summary[2] credits Leon, a student of the Platonic school, with its invention.

While there are divided opinions as to whom belongs the credit of these various methods, it is certain that they were systematized and practiced in Athens during the time of Plato.

Regarding the general characteristics of the Greek elementary geometry, two stand out prominently: (1) It was essentially deductive. Undoubtedly this was to a large extent the cause of mechanics being expelled from geometry. (2) The restriction as to the use of compasses and rule necessarily divided the subject-matter of geometry; so the conic sections did not find a place in common with the geometry of the straight line and the circle. Hence in Euclid's "Elements" we find only

[1] Gow, p. 176; also see Cantor, I, p. 207.

[2] See above, p. 14.

a treatment of the latter geometry. This division exists to-day in the elementary field, although it has been the custom for some authors to put a synthetic treatment of the conics at the end of the text after the geometry of solids. As for methods of attack in elementary geometry, the Greeks invented all those that are in common use to-day. We should specially mention the method of analysis and the reductio ad absurdum.

CHAPTER II

THE WORK OF EUCLID AND HIS INFLUENCE ON THE SUBSEQUENT TEACHING OF GEOMETRY[1]

THE ELEMENTS OF EUCLID

As has been mentioned, Euclid (cir. 300 B.C.) compiled and arranged in an orderly manner the work of his predecessors. Proclus, after quoting from the Eudemian Summary, closes with these words,[2] "Those who have written the history of geometry have thus far carried the development of this science. Not much later than these is Euclid, who wrote the 'Elements,' arranged much of Eudoxus' work, completed much of Theætetus's, and brought to irrefragable proof propositions which had been less strictly proved by his predecessors."

Before considering the special features of this work, let us summarize the main contributions to Euclid according to the opinion of Dr. Allman.[3] He holds that after the Pythagoreans, Euclid was indebted most to Eudoxus and Theætetus. To the Pythagoreans he attributes the substance of Books I, II, and IV, the doctrine of proportion and of similar figures "together with the discoveries respecting the *application*, *excess*, and *defect* of areas—the subject matter of the sixth book: the theorems arrived at, however, were proved for commensurable magnitudes only, and assumed to hold good for all." To Eudoxus is credited

[1] For a readable account of the work of Euclid and his later influence, see Frankland, *The Story of Euclid.* The relation of Euclid's "Elements" to later texts and progress in geometry is set forth by Professor Gino Loria in his *Della varia fortuna di Euclide.* In lighter vein, one may read an entertaining controversy over the merits of Euclid by turning to Dodgson, *Euclid and his Modern Rivals.*

[2] Gow, p. 137.

[3] *Op. cit.*, p. 211.

Books V and VI, and the bulk of Book XII. To Theætetus is given a part of Book X and Book XIII. Euclid also is given the credit of contributing to Book X. We cannot take Allman's judgment as final, but his conclusions show how little Euclid contributed to the subject-matter of geometry. The fourteenth book, which has been assigned to Euclid, was written about 150 years later by Hypsicles. He has been credited with adding also the fifteenth book, but the opinion is gaining ground that this was written by Damascius of Damascus (cir. 490 A.D.).[1]

The original text which bears Euclid's name consisted of thirteen books. The sequence of subject-matter in these books is as follows:[2] Books I and II are on the geometry of straight lines and areas. Book III treats of circles. Book IV, of regular figures. Book V, the theory of proportion for all kinds of magnitudes. Book VI, the application of this theory to plane figures. Books VI, VIII, and IX treat of the arithmetical theory of proportion. Book X is on arithmetical characteristics of the divisions of a straight line. Books XI and XII treat the geometry of solids, and Book XIII is on the five regular polyhedra.

Book I is introduced by the definitions of point, line, etc. Here are given five postulates and five common notions (later called axioms). One of the postulates deserves our attention, for it represents the points of departure of so-called non-euclidean geometries. It is, "If two straight lines are cut by a third straight line so as to make the sum of the interior angles on the same side of the transversal less than two right angles, then these two lines, if produced, will meet on that side." Euclid, being unable to prove this, had to assume its truth. Playfair (1795) stated the equivalent of this in the form, "Two intersecting straight lines cannot both be parallel to the same straight line." The denial of this led to the non-euclidean geometries of Bolyai and Lobachevski.

The lack of correspondence between the historic development of geometry and its sequence as shown in Euclid's "Elements" has already been referred to.[3] It has been shown, for

[1] Gow, pp. 272, 312-313.

[2] See Heiberg, *Euclidis Elementa*, Leipzig, 1883. Also De Morgan's article, *Eucleides*, in Smith's Dictionary of Greek and Roman Biography and Mythology.

[3] See p. 16 above.

example, that the geometry of circles, the theory of proportion, and the study of the regular solids were developed in an order entirely different from their sequence in the "Elements." Euclid put solid geometry after plane geometry, although the former was partially developed before the latter had been systematized or even fully developed. The systematizing of definitions, axioms, and postulates was not accomplished until Plato and Aristotle turned their attention to this fundamental part of geometry.[1] Euclid was not concerned with the subject-matter of geometry from the standpoint of its growth, his task being to organize this material so as to get a minimum of logical friction. The sequence set by Euclid has to a large extent been studiously copied by his followers.

Another thing peculiar to Euclid was the elimination of all practical work. We have observed that it was largely due to Plato that mechanics was divorced from geometry. Euclid simply reflected the traditions of the Greek people. Geometry was to be evolved from within itself, so there was not even mensuration in the "Elements." Even the quadrature of the circle was omitted. Furthermore, there were no original exercises such as we have to-day in our geometries. All the propositions were worked out in full. It was a book to read, not one to develop, as the term is used to-day. In short it was a philosophical treatise intended for mature minds.

One further feature of Euclid should be mentioned. Hypothetical constructions were not permitted. Thus the theorem, "If a triangle have two sides equal, the angles opposite these sides are equal," could only be proved after it was shown how to construct an isosceles triangle. Hence Euclid begins his first book with constructions. This in itself is of pedagogical value, but the method was carried to the extreme. Modern usage allows these hypothetical constructions.

Regarding the nature of theorems and problems, Euclid makes no distinction in his naming. He calls them all propositions.

Euclid's predecessors had developed the various methods of at-

[1] Gow, p. 176. For an instructive article on the mathematical contributions of Aristotle, see Heiberg, *Mathematisches zu Aristoteles*, in Abhandlungen der Geschichte der Mathematischen Wissenschaften, v. 18, pp. 1-49.

tack already mentioned, but had organized no systematic plan by which all propositions were subjected to an orderly method of proof. The "Elements" bears evidence of such a plan, and to Euclid we are thus indebted for:

1. The general enunciation of the proposition.
2. The particular statement.
3. The construction.
4. The proof.
5. The conclusion.
6. The affixing at the end the Q.E.D. or the Q.E.F.[1]

Some features of Euclid are adversely commented upon by De Morgan.[2] The substance of three of his criticisms, which seem to be well founded, are: (1) Euclid makes no distinction between propositions which require demonstration and those which a logician would see to be nothing but different modes of stating a preceding proposition. Thus the statement, "Everything not A is not B" is equivalent to "Every B is A," but Euclid does not recognize this equivalence. (2) He fails to employ generalizing notions in certain cases. Thus in defining an angle as the sharp corner between two lines that meet, he does not consider the straight angle and the reflex angle. (3) He neglects the formal accuracy with which translators have endeavored to invest the "Elements." He refers to theorems in an indirect manner, either reasserting without reference, or saying "it has been demonstrated." Also, he places theorems among definitions, makes assumptions that are not in the postulates, and omits some necessary proofs. According to De Morgan, Euclid has been considered so perfect that later writers, thinking that they were restoring the original perfection of the book, were, as a matter of fact, improving on the "Elements." Simson is one of these to whom De Morgan refers.

To summarize briefly, Euclid is to be considered the compiler and not the composer of the "Elements." Very little that was original is attributed to him. His great work was to systematize the logic of geometry. This applies not only to the sequence of propositions, but to the orderly arrangement of proof in the propositions. As we shall refer repeatedly to Euclid

[1] Proclus, ed. Friedlein, pp. 203, 210. Compare Heiberg's *Euclidis Elementa*. See Gow, p. 199.

[2] *Op. cit.*, p. 71.

and Euclidean geometry, three characteristic features of the book should be re-emphasized. They are:

(1) Hypothetical constructions are not admitted.

(2) All practical work is excluded.

(3) All constructions are by means of straight edge and compasses only. This bars out the conic sections.

THE LATER INFLUENCE OF EUCLID

After Euclid, mathematics became more practical at Alexandria, due principally to the efforts of Archimedes, Hipparchus, and Heron of Alexandria. But the study of theoretical elementary geometry was kept alive although its subject-matter was little increased. Beginning with Theon of Alexandria, the father of Hypatia, the "Elements" of Euclid began a long series of editions. Theon, who lived in the fourth century A.D., lectured on mathematics at the University of Alexandria, where Euclid had been the first professor. There can be but little doubt that during the interim between Euclid and Theŏn the study of the "Elements" had been carried on, for the interest was such 675 years after Euclid that Theon prepared an edition of the "Elements" for his classes. This edition made some slight changes in the original, and added some commentaries, but the work as a whole was kept intact.[1] Before the time of Theon, the "Elements" had become known throughout Greece, including Asia Minor and the Italian colonies.[2] Indeed, in Italy, although geometry there was essentially practical, Euclid was not unknown, for Boethius (cir. 500 A.D.) incorporated in a work a statement of the propositions of Euclid I plus some others from the second, third, and fourth books, giving at the last the proofs of the first three propositions of Book I.

When Alexandria was destroyed by the Arabs, in 640 A.D., Greek learning found a home in the Syrian cities on the east coast of the Mediterranean. From these schools the Arabs of Bagdad gained something of Greek learning, and the works of Euclid, Archimedes, Apollonius, and Ptolemy were translated into the Arabic. Euclid was partially translated in the time of Harun al

[1] De Morgan, *op. cit.*, p. 68.

[2] *Ibid*

Raschid (786–809), and completely so under Al Mamun.[1] The Arabs are to be considered the preservers of Euclid. They contributed nothing to the science of geometry, their achievements being in arithmetic, trigonometry, and algebra. When they conquered Spain (747), they brought their Euclid with them, but it was nearly 400 years later that this learning was given to Christian Europe.

The Moors guarded their learning very jealously, but in 1120 an English monk, Adelard of Bath, while studying in Spain, succeeded in getting a copy of the "Elements" and translated it into Latin. Another translation was made by Gherardo of Cremona in about 1186, and in 1260 Johannus Campanus made a copy of Adelard's translation and gave it out as his own. These translations had a stimulating effect on the study of geometry. Leonardo of Pisa, in 1220, wrote the first original work on mathematics in Europe that was based on Euclid, Archimedes, and Ptolemy.[2]

In the curricula of the universities, which began their existence at this time, Euclid and Aristotle found a place, but it was only in the German institutions that any great attention was given to the study of the "Elements." In Italy especially the study of Euclid was associated with astrology.[3] We thus see that up to the time of the invention of printing, the study of Euclid was a rather formal affair. The influence of the "Elements" was not far reaching. But with the invention of printing it became possible for the work to be read by a greater number of people, and its use became more common in the higher institutions of learning.

Some of the important editions through which the "Elements" passed should be mentioned.[4] The first edition to be printed was in Latin from the Adelard-Campanus translation

[1] Hankel, pp. 231-234; Sédillot, *Matériaux pour servir à l'histoire comparée des sciences mathématiques chez les grecs et les orientaux*, Tome I, p. 377.

[2] Hankel, pp. 342-348. See below, p. 48

[3] For a more detailed account of the teaching of Euclid in the early universities, see below, pp. 51-53.

[4] For a fairly complete list, see De Morgan's article, *Eucleides*, in Smith's Dictionary of Greek and Roman Biography and Mythology. Also see Heiberg, *Litterargeschichtliche Studien über Euklid*; and Kästner, *Geschichte der Mathematik*, I, pp. 279-398. The latter work is hereafter referred to as Kästner.

from the Arabic. It comprised the full fifteen books, and was printed by Ernest Ratdolt at Venice in 1482. Other editions followed, until in 1509 appeared the one by Paciuolo, who was the first to print a work on algebra. The fifth edition was printed at Paris in 1516, under the title "Contenta." De Morgan says, "From it we date the time when a list of enunciations merely was universally called the complete work of Euclid." This idea was quite common during the sixteenth century. The above edition contained, besides the enunciations of the theorems, the commentaries of Campanus, Theon, and Hypsicles. The idea was prevalent that Theon had supplied the proofs, which Euclid had failed to insert.[1] This is shown by a statement in Xylander's German translation (Basel, 1562), where the reader is warned that the demonstrations were added "nit von jme dem Euclid selbs," but by other learned men, Theon, Hypsicles, Campanus, etc.[2] These five editions appeared within thirty-four years, which shows a revival of interest in the "Elements." Then began the period of printed translations into Greek. The first was by Simon Grynæus at Basel, in 1533. In 1551, Robert Recorde published his "Pathway of Knowledge," which contained the enunciations of the first four books of Euclid, but not in Euclid's order. The first translation of the complete "Elements" into English was by Henry Billingsley, in 1570. The next important English edition was by Robert Simson in 1756. In 1795 appeared Playfair's Euclid, which was a departure, as solid geometry was added from other sources. But the first radical departure from Euclid that was generally adopted was the text of Legendre, which appeared in 1794.

The extent to which the various countries have adhered to the geometry of Euclid will be considered later.[3] This preliminary survey suffices to show us that the influence of the "Elements" has been most enduring since its introduction into the curriculum of the early universities.

[1] Gow, p. 200.

[2] Heiberg, *Litterargeschichtliche Studien über Euklid*, p. 175.

[3] See especially Chapters V and VI.

CHAPTER III

THE TEACHING OF GEOMETRY FROM EUCLID TO THE RISE OF THE CHRISTIAN SCHOOLS

AT ALEXANDRIA

The "Elements" of Euclid marked the culmination of the development of elementary geometry. From a logical standpoint, the system was now complete, so no new developments could be expected in this direction, but there was a wide field for the applications of geometry. Solid geometry was not yet fully developed, and in the field of deductive mathematics there was yet the development of the geometry of the conic sections. To these other fields the later Alexandrian mathematicians turned their attention. In treating that which is contribution to subject-matter, we shall discover some facts as to method that have a particular bearing on our subject.

The first geometer after Euclid was Archimedes of Syracuse (b. 287 B.C.). Although he probably did not live at Alexandria, his writings show a thorough acquaintance with the mathematical knowledge of the earlier Alexandrian school. Archimedes was famous for his applications of geometry to science. The nature of his contributions is best seen from the titles of some of his works.[1] "Equiponderance of Planes," "The Quadrature of the Parabola," "On the Sphere and the Cylinder," "On The Measurement of the Circle," "On Spirals," "On Conoids and Spheroids," "On Floating Bodies." He also wrote a treatise on the half-regular polyhedra and his addition to the geometry of the three round bodies was considerable. Two of his propositions, at least, are well known. He proved that the area of a spherical surface equals four times the area of a great circle, and secondly, that the volume and surface of a sphere equal two-

[1] Archimedes, *Opera omnia*, ed. Heiberg.

thirds the volume and total surface of the cylinder in which it is inscribed.[1] The value of π obtained by Archimedes is so close an approximation that even to-day it is used in ordinary work. He expressed it in the form, "A circle has to the square on its diameter the ratio 11 : 14 very nearly."[2]

In methods of attack, Archimedes continued the use of analysis and exhaustions as begun at Athens. The process of exhaustion as applied to the quadrature of the parabola deserves special mention, as it laid the foundations of the integral calculus.[3] Archimedes frequently gave mechanical proofs for some of his propositions. For example, in effecting the quadrature of the parabola, he gave both a geometric and a mechanical proof.[4] This is of interest pedagogically, for in recent years we hear of efforts being made to make the teaching of geometry more experimental. In the above case we see some historic basis for modern "laboratory methods" in the teaching of geometry.

In his division of subject-matter, Archimedes drew no hard and fast line between the geometric, the arithmetical, and the mechanical. For example, the above proposition occurs in his "Quadrature of the Parabola," but the physical principles on which it is based are found in his Book I of "Equiponderance of Planes or Centers of Plane Gravities." So we see here a recognition of the unity of the mathematical sciences, a principle that we are too far from fully recognizing to-day.

Concerning the work of Archimedes, then, we can say that he added to the subject-matter of solid geometry, he placed physical

[1] According to the wish of Archimedes, a cylinder with its inscribed sphere was engraved on his tomb. Plutarch, *Marcellus*, 17.

[2] From this $\pi = 3^1/_7$ approximately.

[3] For an outline of the plan, see Gow, pp. 226-227. Archimedes' method as explained by him in a letter to Eratosthenes is found in the newly discovered MS. mentioned above. See note, p. 16.

[4] In his Quadrature of the Parabola (prop. 6), a proposition is proved mechanically as follows: ΑΓ is a lever with Β its mid-point. A right-angled triangle ΒΔΓ is suspended from ΒΓ, having ΒΓ equal to one-half the length of the lever, the right angle being at Β. An area Z is suspended from A, and balances the triangle. It is proved that the area of Z equals one-third the area of the triangle. The center of gravity of the triangle has already been determined in the Equiponderance of Planes (I, 14). Other proportions are proved according to the same methods. Archimedes, *Opera omnia*, Heiberg ed.

science on a mathematical basis, and in turn made mathematics practical. He employed the geometric methods of attack already existing. His method in the large showed that he recognized the unity of the mathematical subjects.

It has already been stated that the conic sections were excluded by the Greeks from the domain of elementary geometry. Only those constructions were allowed in the "Elements" which could be effected by means of compasses and the straight edge. Menæchmus (b. cir. 375 B.C.) invented the geometry of the conic sections, but it was Apollonius of Perga (b. 260 B.C.) who systematized the work and put it on a scientific basis. While the invention of this new geometry has certainly had an influence on the teaching of elementary geometry, the significant thing for us here is that the geometry as defined by Euclid and that by Apollonius were not shaped into one coherent system. A complete treatise on the conic sections would include the geometry of the straight line and the circle, two special kinds of conics. But this generalizing treatment was not made then, nor since, in the realm of synthetic geometry. It has been only since the invention of analytic geometry by Descartes that this treatment has found recognition.

The geometry developed at Alexandria after Apollonius was confined almost entirely to the practical. Under Eratosthenes (b. 276 B.C.), Hipparchus (b. 180 B.C.), and Claudius Ptolemæus (b. cir. 87 A.D.), geometry found an application in astronomy. Surveying was put on a scientific basis by Heron (b. 125 B.C.),[1] and extended later by Sextus Julius Africanus (cir. 200 A.D.). By both of these the measurements of heights and distances were emphasized, as was the case in the Italian practical geometries up to the middle of the seventeenth century. The formula for the area of a triangle in terms of its sides is due to Heron. During this period, trigonometry was developed with respect to its applications by Hipparchus and Ptolemy (Claudius Ptolemæus).

Little was added to the subject-matter of geometry during this later period. With the practical completion of solid geometry by Archimedes, there seemed little else to add. We learn that the subject of isoperimeters was studied by Zenodorus (cir. 150 B.C.), who wrote a treatise on this important branch of geometry. The geometry of the sphere was somewhat further

[1] There is some doubt regarding the dates to assign to Heron.

extended by Menelaus (cir. 100 A.D.). His "Sphæra," in three books, is a treatise on spherical triangles. His treatment corresponds in a sense to Euclid's treatment of plane triangles. For example, he proves: In every spherical triangle the sum of two sides is greater than the third (I,5); The sum of the three angles is greater than two right angles (I,11); Equal sides subtend equal angles and the greatest side the greatest angle (1, 8, 9); The arcs which bisect the angles meet in a point (III, 9). The importance of one of his theorems (III,1) to modern geometry is pointed out by Chasles. He proved the theorem both for plane and spherical geometry. It is, "If the three sides of a triangle be cut by a straght line, the product of three segments which have no common extremity is equal to the product of the other three." Concerning the importance of this theorem, Chasles writes, "The proposition in plane geometry of which we shall speak below in the article on Ptolemy . . . . has acquired a new and great importance in modern geometry, where the illustrious Carnot has introduced it, making it the base of his theory of transversals."[1] A century earlier Theodosius wrote a complete treatise on the sphere in three books, but he added little to what was already known. One of his theorems, (I, 13) was, "If in a sphere a great circle cut another circle at right angles, it bisects it and passes through its poles." The converse was also proved.[2] Another important theorem was added to solid geometry by Pappus, who lived at the end of the third century. He proved that the volume of a solid of revolution equals the product of the area of the generating plane figure by the circumference of the circle generated by the center of gravity of the figure thus revolved.[3]

That interest in the "Elements" of Euclid itself did not die out at Alexandria is shown by some of the commentaries upon it. According to Tannery,[4] Heron wrote a commentary. Cantor,[5] however, doubts this. Pappus in his voluminous writings discussed some of Euclid's propositions. Theon (cir. 370 A.D.), who wrote an edition of the "Elements," added much to it bv

[1] Chasles, *Aperçu historique*, p. 26.

[2] Theodosius, *Sphœrica*, ed. Barrow.

[3] Pappus, *Collectio*, ed. Hultsch, p. 682. This is known as Guldin's rule

[4] *La géométrie grecque*, p. 166ff [5] Cantor, I, p. 354.

way of commentary. Finally, Proclus (b. 412 A.D.) wrote a commentary on Book I.[1] We must not forget the work of Hypsicles (cir. 180 B.C.), who wrote the fourteenth book of Euclid, also Geminus (cir. 70 B.C.), who included much valuable historic material in his "Arrangement of Mathematics," and Damascius of Damascus (cir. 490 A.D.), who is thought to have written the fifteenth book.

Claudius Ptolemæus (cir. 139 A.D.) wrote on pure geometry. Proclus[2] (pp. 362–368) has preserved extracts from this work in which it is shown that Ptolemy was not satisfied with Euclid's axiom of parallels and so proposed a proof for the same. This inaugurated the long series of futile attempts to prove this axiom of Euclid.[3]

The treatment of subject-matter by those interested in the practical side of geometry is of significance to us on the method side. It has already been shown that Archimedes tended to unify the various branches of applied mathematics. Heron was the first to carry over geometric symbolism into algebraic operations.[4] "He is the first Greek writer who uses a geometrical nomenclature and symbolism, without the geometrical limitations, for algebraical purposes, who adds lines to areas and multiplies squares by squares and finds numerical roots for quadratic equations."[5] The fact that Heron placed the exercises on heights and distances[6] in his Stereometry II is of historic interest. In many of the later Italian practical geometries we find just this arrangement, and to-day in our sequence of mathematical subjects, it is common to place trigonometry (which grew out of such mensuration) after solid geometry in the school curriculum. There are certainly strong reasons for thinking that we have been following the example set us by Heron of Alexandria.

Stated briefly, the subject-matter of elementary geometry was

[1] Friedlein edited this in 1873. There is an English translation by Thomas Taylor written in 1792. See also Frankland, *The First Book of Euclid's Elements with a Commentary*, 1905.

[2] Cantor, I, pp. 395-396; Gow, pp. 300-301.

[3] For a history of the theory of parallel lines, see Stäckel und Engel, *Die Theorie der Parallellinien von Euclid bis auf Gauss*, pp. 31-135.

[4] Algebra as a science had not yet been developed.

[5] Gow, p. 285.

[6] Cantor, I, p. 363; Gow, p. 281.

enlarged but little by the followers of Euclid at Alexandria. In plane geometry, the subject of isoperimeters was further developed, and solid geometry received practically its present form. But above all, the important work of this period was the development of the theory of the conic sections and of the geometry of measurement. The first of these has never been unified with elementary geometry; the second has, although in varying degree in the different countries and institutions.

THE ORIENTALS

Little was added to geometry by the Hindus. Just how much was original with them it is hard to say. We find Brahmagupta[1] (b. 598 A.D.) using Heron's formula for finding the area of a triangle. He was also familiar with the Pythagorean proposition (Euclid I, 47). One feature of his work was his distinguishing between approximate answers and exact answers. He called the first gross answers. Brahmagupta thus takes $\pi = 3$, thereby giving the gross value of the circumference of a circle, and in a problem following he takes $\pi = \sqrt{10}$, giving what he calls an exact value for circumference and area.[2]

The method of proof used by the Hindus was probably characterized by its brevity. In fact, Aryabhatta (b. 476 A.D.) would only state the theorem, add the figure, and then write "behold!"[3] We know that Aryabhatta, among other Hindu writers, wrote on mathematics in verse. Whether geometry was taught by means of rhymes, we do not know, but the study of geometry would have offered splendid opportunity for such a practice wherever learning by heart was encouraged.

The Arabs contributed even less than the Hindus. They were influenced on the one side by the Hindus, on the other by

[1] Colebrook, trans. of *Algebra with Arithmetic and Mensuration*, from the Sanscrit of Brahmagupta and Bhascara, pp. 295-318. Also see Cantor, I, pp. 605-614.

[2] The area of a triangle whose sides are 13, 14, 15 is found by getting the half sum of 13 and 15 and multiplying by $\frac{1}{2}$ of 14 or 7. In the case of the isosceles triangle whose sides are 10, 13, 13, Brahmagupta multiplies 13 by $\frac{1}{2}$ of 10. The exact values are also given by using Heron's formula.

[3] Fink, *A Brief History of Mathematics*, trans. by Beman and Smith, p. 215. See Rodet, *Leçons calcul d'Âryabhata*

the classical learning of the Greeks. The Arabs have already been mentioned[1] as the preservers of the mathematical learning of Alexandria. We recall that among other works Euclid's "Elements" was translated into the Arabic at Bagdad and was introduced into Europe by way of Spain.

THE ROMANS

The Roman mind was concerned with the practical. The youth was trained in oratory that he might make use of it for practical ends. So it was with mathematics, the end was practical. The geometry of the Romans was associated with surveying and the engineering of warfare. It is known that Julius Cæsar caused a survey to be made of the Roman Empire.[2] For our own system of land surveying we are indebted to the Romans. According to Cantor,[3] the temple-fields of the Etruscans were truly orientated. How this was done is not known, but the Romans later knew how to lay out meridians. Thus Vitruvius, an architect of the time of Augustus (cir. 15 B.C.), and Hyginus, a surveyor of the time of Trajan (cir. 100 A.D.), knew two methods of doing this.[4] On account of these practical interests of the Romans, their mathematical writings were chiefly on mensuration and surveying. The writings of some of them are collected in a work known as the Codex Arcerianus, the contributions being fragments of the works of Frontinus, Hyginus, Balbus, Nipsus, Epaphroditus, and Vitruvius Rufus, all of whom lived during the first two centuries of the Christian era.[5]

[1] See above, p. 32.

[2] Cantor, *Die römischen Agrimensoren und ihre Stellung in der Geschichte der Feldmesskunst*, p. 75.

[3] *Ibid.*, pp. 65-66.

[4] One of these methods was as follows: Let AC be a stake set upright in the ground. At a certain time in the forenoon the shadow will be represented by a line such as BC. With C as a center and BC as a radius draw a circle. Let CD be the position of the shadow in the afternoon when its extremity just touches the circumference. BC = CD. Join B, D and draw the perpendicular bisector of BD. This is the required meridian line.

[5] Cantor I, p. 513ff; Günther, *Geschichte des mathematischen Unterrichts im deutschen Mittelalter bis zum Jahre* 1525, p. 115. Hereafter referred to as Günther. The "Codex" was discovered in 980 by Gerbert, who became Pope Silvester II. See Gow, p. 206.

The impulse to develop this practical geometry came certainly from the nature of their own needs, but the Romans were undoubtedly influenced by the practical geometers of the Alexandrian school. The work of Heron, who developed surveying at Alexandria, was known to them. Archimedes (b. 287 B.C.), we recall, passed the greater part of his life at Syracuse in Sicily. We know that his writings were made generally known to Latin Europe from the Arab translations carried into Spain, but the Roman architect Vitruvius Pollio knew of his work,[2] and it is safe to suppose that use was made of it.

One might expect that Euclid would have found its way into Italy by way of Sicily. The practical geometry of Alexandria did, why not the theoretical? Because the Romans were not interested in that side. But we have evidence that knowledge of the subject-matter and method of Euclid was not unknown to the Romans during the first century of the Christian era. The following passage from Quintilian[3] will show the truth of this: "Order, in the first place, is necessary in geometry; and is it not also necessary in eloquence? Geometry proves what follows from what precedes, what is unknown from what is known; and do we not draw similar conclusions in speaking? Does not the well known mode of deduction from a number of proposed questions consist almost wholly in syllogisms? Accordingly you may find more persons to say that geometry is allied to logic, than that it is allied to rhetoric. . . . Besides of all proofs, the strongest are what are called geometrical demonstrations; and what does oratory make its object more indisputable than proof? Geometry, often, moreover, by demonstration, proves what is apparently true to be false. . . . Who would not believe the asserter of the following proposition: 'Of whatever places the boundary lines measure the same length, of those places the areas also, which are contained by those lines, must necessarily be equal?' But this proposition is fallacious; for it would make a vast difference what figure the boundary lines may form; and historians, who have thought that the dimensions of islands are

[1] Cantor, I, p. 515.

[2] Vitruvius relates how Archimedes came to consider the laws of floating bodies. Vitruvius, *Architecture*, ed. Newton, IX, 3.

[3] *Institutes of Oratory*, Book I; ch. X, 37-41, trans. Watson.

sufficiently indicated by the space traversed in sailing round them have been justly censured by geometricians. For the nearer to perfection any figure is, the greater is its capacity; and if the boundary line, accordingly shall form a circle, which of all plane figures is the most perfect, it will embrace a larger area than if it shall form a square of equal circumference. Squares, again, contain more than triangles of equal circuit, and triangles themselves contain more when their sides are equal than when they are unequal. . . .". The method of geometry was, for Quintilian, of practical value in making more perfect the art of oratory.

We thus see that the subject-matter of geometry under the Romans was of a practical nature. The logic of geometry was of value inasmuch as it was an aid in oratory. Only in one way has the work of the Romans influenced the later teaching of geometry. The later Italian practical geometries under the influence of both the practical work of the Romans and that of Archimedes and Heron, kept alive the interest in applied geometry. As a rule, owing to the standard set by Euclid, it was not combined with the theoretical.

CHAPTER IV

THE TEACHING OF GEOMETRY FROM THE RISE OF THE CHRISTIAN SCHOOLS TO THE YEAR 1525

THE CHRISTIAN SCHOOLS OF THE MIDDLE AGES

Already before the light of Greek learning had been extinguished at Alexandria a new sort of education had sprung up there. The doctrines of Christianity came in conflict with Greek thought, and Christian leaders saw that if Christianity was to attain any success in this competition, its teachers must be trained in the Greek learning.[1] So there arose at Alexandria the catechetical schools, and out of these grew the episcopal schools of later times. All these schools had as their chief function the training of their members for the priesthood. So it is not surprising that the study of mathematics was neglected.

Although Christian education began at Alexandria, the little geometry taught in the schools was derived, not from the Alexandrians, but from the later Roman writers, Martianus Capella (cir. 420 A.D.), Boethius (cir. 480-524), and Isidore of Seville (cir. 570-636). The books written by these men were the great text-books of the Middle Ages up to the thirteenth century.[2] They contained, however, but little geometry. It will be shown

[1] Monroe, *Text-book in the History of Education*, p. 233.

[2] Laurie, *The Rise and Early Constitution of Universities*, pp. 37-38: Günther, pp. 1-2.

About the beginning of the first century B.C., Varro, a Roman, wrote on grammar, rhetoric, dialectic, arithmetic, geometry, astronomy, music, philosophy, and other branches. See Davidson, *The Seven Liberal Arts*, in the Educational Review, 2, p. 469. This also appears in his *Aristotle and Ancient Educational Ideals*, appendix. See also Parker, *The Seven Liberal Arts*, in The English Historical Review, vol. V, p. 431.

later that the geometry of Boethius, as a text, was lost to Europe until a copy was found by the great Gerbert (cir. 980). The work of Capella[1] is embodied in an encyclopædia of nine books, the sixth book treating of geometry, which is more properly geography. This part of the work was followed by definitions of lines, figures, and solids, then the most necessary "demands," all according to Euclid and using the Greek terminology. The geometry of Boethius[2] (cir. 480–524) in like manner is part of a larger work. It begins with definitions, postulates, and axioms. After these follow Book I of Euclid and selected propositions from Books II, III, and IV, but the propositions are merely stated. It is only at the end that any proofs are given, and then only for the first three propositions of Book I. After further definitions and a discussion on the operations with numbers, there is inserted the much discussed chapter on the Gobar numerals.[3] The rest of the geometry is concerned with the mensuration of various geometric figures, most emphasis being laid on the simplest plane figures.[4] As in the geometry of Capella there is no surveying. The geometry of Boethius shows on the whole a tendency to follow the practical geometry of the Romans, but the attention given to Euclid marks it as different from any European geometry up to the time of the introduction of Euclid into Latin Europe by way of Spain.

Two other names are associated with the early Christian education of the Middle Ages, those of Cassiodorus (cir. 480–575) and Isidore of Seville (cir. 570–636). Both of these recognized the order of the trivium and quadrivium in the arrangement of the subject-matter in their writings.[5] Under the trivium were grammar, rhetoric, and dialectics; under the quadrivium, arithmetic, music, geometry, and astronomy. These constituted the seven liberal arts of the Middle Ages. But the geometry

[1] *De nuptiis philologiæ et septem artibus liberalibus. Libri novem.* Edition 1539. See Cantor, I, pp. 527-528; Kästner, I, p. 251.

[2] *De Institutione arithmetica, musica, geometria*, ed. Friedlein, p. 373ff.

[3] Some claim that this treatment of the Gobar (dust) numerals is not authentic. See Weissenborn, *Die Boetius Frage.*

[4] Some of the examples are not worked correctly. For example, Boethius finds the area of an equilateral triangle whose side is 30 to be 390, while for one whose side is only 28 the area is 406.

[5] Cantor, I, pp. 529-532; 772-775.

in these writings was, like that of Capella, closely related to geography.

Up to the time of Gerbert (d. 1003), we do not find that instruction in geometry went beyond the learning of definitions and rules, and the making of a few simple constructions.[1] The subject was generally taught in the church schools,[2] but it was given stepmotherly care because geometry was of no practical significance for those studying for the clergy.[3] There seems to have been no use of the Roman practical geometry of the time of Vitruvius. The study of geometry received no greater attention in the Palace School of Charles the Great, where, under the direction of Alcuin (735-804), the subject was taught as one of the seven liberal arts.[4]

After Gerbert, the teaching in the church schools was still based on the medieval writers already mentioned, but there was added a practical phase which carries us back to the geometry of Heron and that of the Roman surveyors (gromatici). Gerbert, who was the head of the cathedral school at Rheims and later became Pope Silvester II, made two discoveries of mathematical treatises that were of importance.[5] The first was the finding of the *Codex Arcerianus*, which contained the works of the Roman surveyors. The second was the finding of a copy of the geometry of Boethius at Mantua. These discoveries have a double significance for us. The fact that Gerbert discovered them means that those works *as such* were unknown in the medieval schools up to that time, and the nature of the subject-matter of the geometry then taught bears out this fact. We recall here that the Codex treated geometry with reference to its application in surveying and the measuring of heights and distances, and the geometry of Boethius the mensuration of ordinary geometric figures only, this being prefaced by some propositions from Euclid. Since the discoveries of Gerbert are considered genuine, one must conclude that the church

[1] Cantor I, p. 113.

[2] Specht, *Geschichte des Unterrichtswesens in Deutschland von den altesten Zeiten bis zur Mitte des dreizehnten Jahrhunderts*, p. 144.

[3] *Ibid.*, p. 143.

[4] Günther, pp. 25-26.

[5] Cantor, I, pp. 797-824

schools were not in possession of the geometry of Boethius, although they were acquainted with his other writings.[1] [2]

The geometry of Gerbert[3] (cir. 980) shows that he was influenced by the practical work of the Codex, but not by the Euclidean feature of the geometry of Boethius. The book begins with definitions, then comes a discussion of units of measure, after which follows a great deal of work on mensuration, more than is given by Boethius. Gerbert also gives problems on finding heights and distances, using the astrolabe and the mirror.[4] Gerbert used some inexact methods in finding areas. Thus in an equilateral triangle whose side is a he finds the area by the formula,[5] Area $= \frac{1}{2}a\ (a - \frac{1}{7}a)$. He takes $a = 7$ and thus finds the area to equal 21. In another place the area of an isosceles trapezoid is found by multiplying one-half the sum of the parallel bases by one of the equal legs.[6] Brahmagupta, we recall, was more particular, for he gave both "gross" and "exact" answers. If we are to accept the opinion of Gow,[7] this tendency to inaccuracy can be traced back to Heron of Alexandria. The particular formula used by Gerbert for the area of an isosceles trapezoid can be traced back even to the ancient Egyptians, for we remember that Ahmes employed this incorrect rule. Although Gerbert's book was decidedly practical, we catch a glimpse of a logical proof where he shows that the angle-sum of a triangle equals two right angles, using the Euclidean method of proof by drawing a line through a vertex parallel to the opposite

[1] Günther in a recent article says that the teaching of geometry in the early Middle Ages was based on the geometry incorporated in the treatises of Capella and Isidore of Seville. No reference is made to the geometry of Boethius. See Günther, *Le développement historique de l'enseignement mathématique en Allemagne*, trans. by Bernoud in Enseignement Mathématique, 1900, pp. 237-264. Hereafter referred to as Günther, Ens. Math.

[2] Weissenborn does not credit Boethius with having written the geometry ascribed to him. See his *Boetius Frage*

[3] For an edition of Gerbert's geometry, see *Opera mathematica*, edited by Bubnov. Also Migne, *Patrologiæ cursus completus*, v. 139, pp. 84-134.

[4] For description, see below, p. 57.

[5] Günther, p. 119.

[6] Bubnov, *op. cit.*, p. 350

[7] *Op. cit.*, p. 284.

side. The three kinds of triangles are employed, however, the acute, right, and obtuse angled.[1] From the pedagogical point of view, the order in which Gerbert places a series of definitions is of interest. He defines solid, surface, line, and point in the order named. Euclid used the reverse order. It is recognized to-day that Euclid's order is not pedagogical, and that the proper method is to proceed from that which is the most familiar in experience, the solid, and arrive lastly at the conception of the point.

As has been said, before Gerbert the geometry in the church schools did not include any practical applications. That the geometry of Gerbert may have had some influence on later teaching is shown from the fact that in some of the church schools there were exercises in finding heights and distances. At the monastery of St. Gall, geometry was studied in the "open." "Die Geometrie wurde nicht nur in der Schulstube, sondern am liebsten im Freien studiert. . . . es wurde die Höhe vom Erdboden bis zum Kirchturmhahn gemessen, oder ein jüngst dem Kloster vermachtes Gut wurde abgesteckt."[2]

To summarize briefly, the teaching of geometry in the church schools of the middle ages was limited to the learning of definitions and the performing of a few simple constructions with rule and compasses. The subject was generally taught but there was no real instruction as we understand it to-day. The influence of Euclid was practically nil. After Gerbert there was a little work also in the finding of heights and distances, the astrolabe and the mirror being the field instruments employed. The method of the recitation was such that an ability to memorize was the main desideratum.[3]

OTHER BOOKS ON PRACTICAL GEOMETRY BEFORE THE RISE OF THE UNIVERSITIES. BOOKS THAT INFLUENCED THE UNIVERSITY TEACHING

Since we are concerned with the practical as well as the logical in the teaching of geometry, it is necessary that we discuss, somewhat briefly, certain mathematical works chiefly of the thir-

[1] Günther, p. 116.

[2] *Ibid.*, pp. 113-114 (ref. Dümmler, Ekkehart, iv, etc., s. 23ff).

[3] *Ibid.*, pp. 79-80

teenth century that influenced the practical mathematics of the early universities and affected, although indirectly, the character of the geometry taught in many secondary schools of the sixteenth and succeeding centuries.

The first of these writers was Leonardo of Pisa, who was the first in Europe to write on algebra, and who is famous also as having spread the knowledge of the Arabic (Hindu) system of numerals in Europe. But it is his "Practica geometriæ,"[1] which appeared in 1220, that has claim for attention here. By this time Euclid had been translated twice into the Latin, but Leonardo's work in its general treatment was not at all Euclidean. It must be classed as a practical geometry as its title indicates, although, as Professor Cantor[2] points out, the stereometric propositions are drawn from Books XI, XII, XIII, and XIV of the "Elements."[3] The book is divided into eight parts preceded by an introduction. In the latter are the usual definitions, and also statements of some important geometric facts given in Euclid as theorems. Thus in considering the propositions involved when two parallel lines are cut by a third line, Leonardo states merely which angles are equal and which are supplementary. The definitions are followed by a discussion of the different units of measurement and their uses, arithmetical and geometric. Part I deals with number operations illustrated geometrically, and also treats proportion. The author shows his acquaintance with Euclid where he draws a circle containing two intersecting chords, forms the equation of the proper products, and adds that it is the same as shown in Euclid. Part II treats square root. Part III contains the mensuration of the various plane figures. Part IV, the various ways of dividing plane figures into equal areas by auxiliary lines. Part V, cube root. Part VI, the mensuration of solids, and Part VII, the mensuration of heights and distances with reference to the use of the geometric quadrans. Part VIII gives further work on the mensuration of circles and their inscribed and circumscribed polygons. We thus see that Leonardo was not concerned with geometry from the Euclidean standpoint.

[1] The "Practica geometriæ" has been edited by Boncompagni (1862). Unless otherwise stated our authority is this edition.

[2] Cantor, II, p. 39.

[3] Cantor states that Curtze holds that Leonardo may have had access to the translation of the "Elements" by Gherardo of Cremona.

But a certain logical consistency is observed in the treatment of his subject-matter. He begins with definitions. As the work is to be concerned with practical geometry, there is next considered units of measure and the operations of arithmetic illustrated geometrically. Proportion and square root precede the work in the mensuration of plane figures. Before the mensuration of solids is taken up, cube root is explained. After the mensuration in three-dimensional space, the author proceeds to the mensuration of heights and distances, the quadrans being the field instrument employed. This work with the quadrans really involves the rudimentary principles of trigonometry,[1] and thus we see that Leonardo, like Heron of Alexandria, placed what was essentially trigonometry after the treatment of solid geometry. In brief, Leonardo of Pisa systematized the subject-matter of practical geometry. His book stood as a type for the later Italian practical geometries.

Other writers on practical geometry during the thirteenth century were Savasorda, Jordanus Nemorarius, and Robertus Anglicus. According to Curtze,[2] Savasorda wrote his "Liber embadorum"[3] before Leonardo wrote his "Practica geometriæ," and the latter work was based on the book of Savasorda. Jordanus[4] was a German mathematician, who wrote extensively on mathematics. The "De triangulis" represents best his contributions to geometry. It is divided into four books. Some of the propositions in the first book show an acquaintance with Euclid. The other books are largely on mensuration and on arcs and chords of circles. The subject-matter adds nothing not in the book of Leonardo.

Another writer on practical geometry during this pre-university period was Robertus Anglicus (cir. 1271), who lived in France. His "Tractatus quadrantis,"[5] as the name would

[1] Leonardo emphasized this work but little, using only three pages for this kind of applied geometry.

[2] Curtze, *Urkunden zur Geschichte der Mathematik im Mittelalter und der Renaissance.* See Part I, *Der "Liber Embadorum" des Savasorda in der Übersetzung des Plato von Tivoli.*

[3] MS. (Latin 11246) trans. Plato of Tivoli in the Bibliothèque Nationale, Paris.

[4] Cantor, II, pp. 73-86.

[5] *Der Tractatus Quadrantis* in Deutscher Übersetzung aus dem Jahre 1477, herausgegeben von M. Curtze, pp. 43-63. It was recently (1897) published in French by M. Paul Tannery

indicate, was concerned with the measuring of heights and distances. There is some mensuration of plane figures and problems on the contents of casks and vats. The work is more confined to the practical than that of Leonardo or even Jordanus.

It is not hard to understand why this interest in practical geometry was kept alive. When Euclid finally received recognition by the universities the aim was only to learn the "Elements," there being little incentive to add to or improve the subject-matter. But there was a wide field for the applications of geometry. As has been seen, this was taught largely in connection with geography and astronomy. Before considering the teaching of geometry in the universities of the Middle Ages, the works of some other writers up to the sixteenth century will be briefly summarized. Some of these influenced the mathematical teaching in the universities more than any of the practical works already mentioned. The "Sphæra" of Sacrobosco (b. 1244) ranks as one of the most noteworthy of these. It was essentially a mathematical astronomy, and was widely used in the universities.[1] The "Geometria speculativa" of Bradwardine. (b. 1390) had a wide influence. Regiomontanus (b. 1436), who received his mathematical instruction from Puerbach at the University of Vienna, added much to practical mathematics in writing his "De triangulis."[2] This was on trigonometry, but included much of the work which had formerly been treated in so-called geometries. Here we see a demarcation, and trigonometry as a science assumed more of an independence from this time on. The Germans, Widmann and Adam Riese, who were contemporaries of Regiomontanus, should be mentioned as having written on practical geometry.

The books that have just been considered, beginning with that of Leonardo of Pisa, show no advance in logical geometry, only traces of Euclid's influence being perceptible. But they stimulated the study of practical geometry, and so made an essential contribution to the development of mathematical science. We shall now turn to the influence of Euclid on the mathematical instruction in the universities.

[1] Cantor, II, pp. 87-91; Suter, *Die Mathematik auf den Universitäten des Mittelalters*, p. 67. Hereafter referred to as Suter

[2] Cantor, II, pp. 254-289.

THE UNIVERSITIES OF THE MIDDLE AGES

The courses of study in the early universities were founded on the seven liberal arts as outlined by Capella and Cassiodorus, and later by Alcuin, and by Hrabanus Maurus at Fulda.[1] So the first instruction in geometry could have been little different from that given in the church schools at that time. But a change came when Euclid finally entered the university curriculum.

It was over 200 years after Adelard of Bath translated Euclid from the Arabic before university regulations demanded that a student must attend lectures on mathematics to be entitled to a degree. Previous to this time, however, lectures on mathematics had been given at some of the newly founded universities. Thus, Sacrobosco (John of Holywood) taught astronomy and mathematics the last years of his life (d. 1256) at the University of Paris.[2] It is not improbable that it was his influence that caused Paris to pass a statute (1336), that unless a student had attended lectures on mathematics he was not entitled to a degree. "From a preface to a commentary on the first six books of Euclid, dated 1536, it appears that a candidate for the degree of M.A. was then required to take an oath that he had attended lectures on the said books."[3] An Oxford statute in the thirteenth century required for the licentiate six books of Euclid.[4] But this was a standard beyond the ordinary reading, for as late as 1450, only the first two books were read.[5] In Prague, founded in 1350, the first six books of Euclid were required for the master's degree.[5] The statutes of the University of Vienna for the year 1389 required one book of Euclid for the bachelor's degree and five books for the licentiate.[6] At Heidelberg, founded in 1385, no mathematics was required for the bachelor's degree during the remaining years of the fourteenth century,[7] and for the licentiate (in 1388) only the first three books

[1] Suter, p. 48.

[2] Cantor, II, p. 87.

[3] Gow, p. 207; See Hankel, pp. 354, 355 and Kästner, I, p. 260.

[4] Suter, p. 64.

[5] Gow, p. 207.

[6] *Ibid.*, p. 77. During the period from 1365 to 1400 lectures were also given on Boethius, which contained some of Euclid. Aschbach, *Geschichte der Universität in ersten Jahrhunderte ihres Bestehens*, p 93.

[7] Suter, p. 79.

of Euclid were read.[1] Cologne (founded in 1388) also did not require any mathematics for the baccalaureate, but three books of Euclid were required for the licentiate.[2] At the University of Bologna, geometry was taught as early as 1383 by Antonio Bilotti of Florence,[2] but it was associated with astrology.[3] The University of Leipzig (founded in 1409) required the "Sphæra materialis" for the baccalaureate, and for the licentiate, Euclid, besides other requirements. The universities of Italy seem to have required less geometry than did the other universities mentioned. The interest there was primarily in astrology. Reference has already been made to this in the teaching at Bologna during the fourteenth century. As late as 1589, in the same university, Galileo taught during two years, the "Elements" of Euclid, the "Sphæræ" of Sacrobosco and of Theodosius, and the "Quadripartium" of Ptolemy, the latter being a work on astrology.[4] We learn also that in 1598 mathematical lectures were given at Pisa on this latter work.[5]

Regarding then the mathematical instruction in the universities of the Middle Ages, it appears that more advance was made in the German institutions. In those of France, England, and Italy, the work was less extensive. The study of Euclid was prescribed at Oxford in the thirteenth century, but it was not until the latter half of the fourteenth century that the study was taken up in any serious way by the various universities. By that time candidates for the master's degree were studying, at the most, the first six books of the "Elements." For the bachelor's degree, during this same period, little or no Euclid was required.

The methods of instruction were essentially the same in the different universities of the Middle Ages.[6] As texts were

[1] Günther, p. 211. [2] Suter, p. 79

[3] Gherardi, *Einige Materialien zur Geschichte der mathematischen Facultät der alten Universität Bologna*, ins Deutsch übersetzt von Maximilian Curtze, p. 20.

[4] Gherardi, *op. cit.*, pp. 14-15.

[5] Hankel, p. 357; Rashdale, *The Universities of Europe in the Middle Ages*, Vol. I, pp. 249-250.

[6] Kaufmann, *Die Geschichte der deutschen Universitäten, Zweiter Band: Entstehung und Entwickelung der deutschen Universitäten bis zum Ausgang des Mittelalters*, pp. 355-356.

very rare owing to the great labor involved in copying by hand, it was the custom of the professors to read from the text while the students took notes. When the reading was interspersed by the commentaries of the professor, these were generally dictated to the students.[1] Sometimes the students read instead the professor's manuscript and copied from it. The method of reading and copying was varied by discussions on some parts of the text. These took the form of disputations, which were usually held once a week. We are to consider that Euclid was taught in conformity with these methods.[2]

[1] At Paris there was a statute against such dictation. Rashdale, *op. cit.*, Vol. I, p. 438.

[2] For accounts of the methods of class instruction in the Universities of the Middle Ages see: Paulsen, *Geschichte des gelehrten Unterrichts auf den deutschen Schulen und Universitäten vom Ausgang des Mittelalters bis zur Gegenwart*, pp. 18-19 et al.; Günther, pp. 192-197; Rashdale, *op. cit.*, Vol. I, pp. 220-250, 436-438, Vol. II, pp. 452-457; Kaufmann, *op. cit.*, pp. 342-370; Compayrè, *Abelard and the Origin and Early History of Universities*, pp. 167-184.

CHAPTER V

THE TEACHING OF GEOMETRY FROM THE YEAR 1525 TO THE PRESENT TIME

The end of the fifteenth and the beginning of the sixteenth century marked an epoch in the history of Europe. In 1453, Constantinople was taken by the Turks and Greek learning found its way into Italy. The scholasticism of the Middle Ages had worn itself out and educated Europe was ready for the intellectual feast awaiting it. The printing-press, which was invented about this time, was a powerful agent in bringing this about. Thus was the Renaissance inaugurated. In 1492, America was discovered and a great stimulus was thereby given to trade. It was at this time that changes were wrought in the aim and work of the schools and the universities that have a bearing on our topic. By the beginning of the sixteenth century, the universities were regularly teaching Euclid. It was natural under the influence of the new education that the universities should extend their work. This meant that the more elementary work should be taken up by other schools. And so by the time of the Reformation we find arising the different institutions under the names of Gymnasien, Pädagogien, Lyceen, etc.[1] About the same time (1525) mathematical instruction reached down into the German Volksschulen.[2] We thus see the beginning of the pressure of the university on the secondary schools, and secondly, that of the secondary on the elementary. This presents a serious problem with us to-day. To understand better the development of the teaching of geometry, we shall first examine some of the early texts that were printed in various countries, especially in Italy, Germany, and France.

[1] Suter, p. 48

[2] Günther, p. II

SOME EARLY PRINTED BOOKS ON GEOMETRY

Up to the time of the invention of printing, two types of geometries had appeared. There was only one kind of logical elementary geometry, Euclid. The practical geometries were of one general standard inasmuch as they were almost entirely independent of Euclid, but they varied somewhat in the sequence and content of their subject-matter. These two types continued after the fifteenth century. The practical geometries began to decrease in number about the middle of the seventeenth century, by which time a third type of geometry began to be common. This type might be called a practical Euclid, a geometry built on the logical lines of Euclid, but recognizing the practical value of the subject.[1]

The first important printed work on mathematics was the "Summa de arithmetica geometria proportioni et proportionalita" (1494)[2] of the Italian Paciuolo, commonly known as Lucas di Borgo. The part devoted to geometry is much like the "Practica geometriæ" of Leonardo of Pisa, which was written nearly 300 years earlier. The figures are drawn in the margin as in the latter work. In both books the drawings are very poor, often inaccurate. In the figures representing solids there is an entire lack of perspective, this being shown particularly in the pyramid, the cone, and the cylinder. Many of the problems in mensuration are identical in both, and the treatment of solids is very much the same.

The following are some of the differences: By the time of Paciuolo the influence of Euclid in southern Europe was being more widely felt, so we are not surprised to find in his geometry some parts of Euclid. Part I of Paciuolo begins with definitions, followed by two chapters corresponding to Books I and II of Euclid. The first book gives no proofs, merely stating the propositions. In this respect we are reminded of the treatment of Euclid in the geometry of Boethius. The second book of Euclid is more fully explained. Thus the construction of the problem on the Golden Section is carefully given. The next chapter, IV, corresponds to Book VI of Euclid, which treats of

[1] This type will be mentioned below in connection with the teaching of geometry in the various countries.

[2] Edition of 1523 referred to here.

proportion applied to plane figures. Paciuolo gives more attention than does Leonardo to the use of instruments for simple surveying. Here we find the quadrans (square form), the plumb quadrans, the use of the staff and shadow, and the mirror for finding heights. When compared with the book of Leonardo, Paciuolo's has two rather prominent features: its recognition of Euclid, and the attention given to field problems.

The next important practical geometry appeared in France in 1556, written by Orontius Fineus.[1] He, like Paciuolo, ranked as a prominent writer on mathematics during the first half of the sixteenth century. It is significant that such writers wrote practical geometries. Euclid of course was considered unchangeable, but the fact that these men and others wrote on practical geometry shows where their real interest lay. As we shall see, these writings had an influence on mathematical teaching.

Fineus abandoned the placing of figures in the margin,[2] putting them in the body of the text. The drawings are generally good. Euclid is not recognized directly, the subject-matter being devoted to mensuration, applied both to the ordinary geometric figures and to field problems. In this latter work, the various instruments in common use in surveying are exemplified.

In our teaching to-day we are beginning to realize that the geometry should be more experimental. The transit and plane-table are being employed by our best teachers in some forms of simple surveying. It is of value to consider the instruments used for such work in the Middle Ages and up to the seventeenth century. The various practical geometries during this period generally described the use of these instruments. The most common instruments were the astrolabe, quadrans, plumb-quadrans, staff (and shadow), "La Croce," the ordinary square, the baculus or Jacob's staff, and the mirror. The astrolabe was used in Gerbert's geometry in finding heights and distances. The Arabs were familiar with it in Gerbert's time, for we learn from Cantor (I, pp. 705–706), that a certain Arab, As-Sâgânî, who died in 990, was a maker of astrolabes. The astrolabe was a circular instrument generally a foot or less in diameter. In using it, say for finding heights, the observer sighted across its center, the instrument being held

[1] *De re & praxi geometrica, libri tres.*

[2] In the practical geometry of Leonardo of Pisa they are in the margin

vertically. Knowing the distance to the object and the readings on the astrolabe, the observer was enabled by proportion to find the required height. The quadrans in its various forms was used in essentially the same way. The mirror was a hemispherical surface used generally for finding heights. It was placed on the ground, and the observer placed himself so as to see the reflection of the object in the mirror. Knowing his own height and two distances, he found by proportion the required height. The use of the square is interesting and could well be employed in school practice to-day. It is required to find the distance to a given object from a given point. A staff is placed on the given point. An ordinary square is placed in a vertical plane with the right angle uppermost on the top or by the side of the staff so that the observer can sight along the longer arm at the distant object. Holding the square fixed, he next sights along the shorter arm, marking on the ground the point determined by this line of sight. The observer knows the height of the staff and the distance from the point just determined to the foot of the staff. By proportion, he finds, as a third proportional, the required distance.

Practical geometries on the lines laid down by Fineus were quite common in Italy in the sixteenth century. A book by Cosimo Bartoli[1] is a close reproduction of the one by Fineus. In many cases the figures and wording are identical. In 1567 appeared the work of Pietro Cataneo,[2] which is concerned with mensuration only. The work of Silvio Belli[3] (1569) deals primarily with the surveying of heights and distances. The book of Gargiolli[4] (1655), which followed the plan of Fineus, shows that as late as the middle of the seventeenth century there was still an interest in books that dealt with the mensuration of plane and solid figures and with the surveying of heights and distances.

Another type of geometry, which illustrates the correlation between algebra with geometry, is represented by the work of Gloriosus[5] (1627). The author works many problems by the

[1] *Del modo di misurare.*

[2] *Le pratiche delle duo prime mathematiche.*

[3] *Libro del misurar con la vista.*

[4] *Iride celeste.*

[5] *Exercitationum mathematicarum.* Decas prima.

aid of algebra, in which he refers to the work of Tartaglia and Vieta of the previous century. In one problem simultaneous quadratics are involved.[1] A book by Gloriosus, which appeared twelve years later, shows that there was current an interest in proving theorems not found in Euclid.[2]

The sixteenth century saw a change in the aim of practical geometry in Germany. Under the influence of Peurbach and Regiomontanus in the fifteenth century, and before them Jordanus Nemorarius in the thirteenth, geometry saw its application largely in the fields of astronomy and surveying. Under the influence of Albrecht Dürer the sixteenth century applications were directed towards architecture and the building arts.[3] Dürer's "Underweysung der Messung mit dem Zirckel" (1525) indicates these lines of application. That the book had an influence outside of Germany is shown in the Italian treatise of Bartoli (1589) mentioned above, where some plans of sections of a cone are given that are almost identical with those in Dürer's work. Various spirals and fanciful designs show the aim in

[1] Some symbolism employed is worth noting. A_q is used for our a^2. The former symbol was first used by Vieta (1600). The symbols for plus and minus are given as $\vdash\!\!+\!\!\dashv$ and $\vdash\!\!-\!\!\dashv$. These are variations of the modern symbols first given in print by Widmann in 1489. The symbol for square root is given as ℞, which can be traced back in print to Paciuolo (1494). No sign is used here for equality.

The symbolism contained in some manuscript notes written in the margin of an edition of Euclid by Sebastian Curtius may be mentioned in this connection. The symbol of equality is written ∞. As Descartes (1637) first employed this sign, and since the book of Curtius was printed in 1618, it is evident that the marginal notes were inserted at least after 1637. The notes further give □ and ▭ as symbols for the words "square" and "rectangle," respectively, used in written explanations just as we find them in our texts to-day. See Curtius, *Die sechs ersten Bücher Euclidis*, 1818. The copy in which the above-mentioned notes are inserted is owned by Professor David Eugene Smith.

[2] *Exercitationum mathematicarum.* Decas tertia. In this is given a controversy over the proof of the theorem that the three altitudes of a triangle are concurrent. The proof finally given by the author is fallacious.

[3] A work on elementary geometric constructions appeared anonymously in Germany in the fifteenth century under the title *Geometria deutsch.* It was the first printed book on geometry in the German language. See Günther, pp. 347-354; Cantor, II, pp. 450-452.

the latter book. We find here a tendency to generalize the notion of angle, curvilinear angles being illustrated.[1] According to Günther,[2] Dürer was the first to define parallel lines as lines that are everywhere the same distance apart. As we shall see, this definition became quite common from this time on. The geometry of the single opening of the compasses, which was first seriously considered by Abul Wafa[3] (cir. 988), was employed by Dürer in an approximate construction of the regular pentagon.

The book of Orontius Fineus and those of the Italian writers of the sixteenth century mentioned above show a strong tendency to hold to the special. The path from the particular to the general is a long and tedious one. To illustrate, they rarely used results from a previous exercise, taking frequently a page to re-explain in detail the most simple processes. The treatment of the mensuration of triangles by Fineus illustrates this still better. Right triangles, subdivided into isosceles and scalene, are first considered. Then acute-angled triangles, divided into equilateral, isosceles, and scalene. The isosceles case is subdivided into the cases where the equal sides are either greater or less than the base, then obtuse-angled triangles are considered, subdivided into isosceles and scalene. The value of an altitude is given in the latter case. In all these cases, where the principle is exactly the same, the complete work is given. Heron's formula is then used, but first on right triangles and then scalene. The same sort of repetition is employed in the treatment of the various quadrilaterals. In modern practice we follow Euclid in treating many of the special forms after the general principle is established. These writers treated triangles from the standpoint of mensuration just as the Greeks before Euclid treated them from a logical point of view.[4]

The illustrations in the books mentioned also have a significance for us. Now that drawings were no longer put in narrow margins, we find frequently a large part of a page given to an illus-

[1] Mercator used these same notions in his geometry over a hundred years later (1678).

[2] *Op. cit.*, p. 162.

[3] Cantor holds that this can be traced back to Pappus and the earlier Greeks. See Cantor, I, pp. 421, 700.

[4] See above, p. 21.

tration which indicates some phase of human activity. This is invariably in the form of some problem in surveying. Here is a picture showing a man in the act of surveying a stream, there one in which the height of a tower is being found. All this makes the book more attractive and is an aid to the reader. This illustrating of a scientific book is a commendable thing. Textbook writers to-day are reminding themselves that a picture on the page is of pedagogical value.

GERMANY

Although the Gymnasia and other secondary schools which were founded in the sixteenth century had as one of their functions the preparation of students for the universities, this did not mean that geometry as taught in the universities was carried over into the curricula of the secondary schools. This transference of the Euclidean system was not fully accomplished for over 200 years.[1] Before the foundation of these preparatory institutions, most of the universities had under their direction schools similar to the above, preparatory to the Faculty of Arts in which instruction was given in Latin, logic, rhetoric, and in arithmetic."[2]

We have already seen that at the end of the fifteenth century the universities demanded at the most the first six books of Euclid for the master's degree. Cantor[3] says that during the first half of the sixteenth century it was customary to read the first five books of the "Elements" in the universities. In 1521, Melanchthon demanded and received a chair in mathematics at the University of Wittenberg. A little later there was one professor for elementary mathematics and one for the superior. Ratke and Reinhold each occupied the first of these two chairs, and the teacher of "Mathesi inferior" discoursed on the elements of arithmetic and geometry.[4]

That the professors in the universities were becoming more

[1] We shall see later that Euclid as such was never generally used in the German schools.

[2] Suter, p. 48.

[3] Cantor, II, p. 394.

[4] Günther, Ens. Math., pp. 237-264; Paulsen, *op. cit.*, pp. 154-155; Hartfelder, *Philipp Melanchthon als Præceptor Germaniæ*, in Monumenta Germaniæ Pædagogica, 7, p. 310.

interested in Euclid is seen from some of their writings. At the University of Vienna, Johann Vögelin, in 1528, wrote his "Elementale geometricum ex Euclidis geometria."[1] At Basel, in 1533, Simon Grynæus the elder published his edition of Euclid with the Commentaries of Proclus. This was the first edition of Euclid printed in Greek.[2] Also at Basel (1562), Xylander printed the first German edition of the "Elements," which consisted of the first six books only.[3]

We may conclude that in the sixteenth century the teaching of Euclid remained practically the same in the Universities. But we observe an increased interest in the translation of the "Elements." By the middle of this century it was translated from the Latin into the Greek, and most important of all into the vernacular, which would indicate an increased interest in its study.

In the sixteenth century very little attention was given to he study of geometry in the secondary schools. Even the practical was not universally taught, this being particularly noticeable in the evangelical schools. Melanchthon assigned a small place to mathematics in the programs of the middle schools.[4]

The Gymnasium at Nuremberg was founded in 1526 on a base half academic. The teacher in mathematics was the celebrated Schoener, the manufacturer of globes. Mathematics was assigned an advantageous place and even when the first success of the school diminished, the classes in mathematics were attended in a satisfactory manner.[5] "This," says Günther, "is comprehensible in a center of traffic and industry." Geometry does not seem to have been taught in the Nuremberg Gymnasium in the sixteenth century. In 1556 the program of the Cathedral School at Wurtemberg provided for "Rechnen" and "Lectio sphærica."[6] No mention is made of geometry. On the other

[1] Cantor II, p. 394.

[2] De Morgan, article, *Eucleides*, pp. 71-72.

[3] According to Cantor (II, pp. 550-551) Scheubel printed some of the arithmetical books of the "Elements" (VII, VIII, and IX). De Morgan (p. 73) gives more than this, adding also books IV to VI inclusive.

[4] Günther, Ens. Math., pp. 237-264.

[5] *Ibid*, pp. 249-250.

[6] Friedrich, *Über die erste Einführung und allmähliche Erweiterung des mathematischen und naturwissenschaftlichen Unterrichts am Gymnasium zu Zittau*, p. 27. Hereafter referred to as Friedrich.

hand, at Strassburg, the program of 1578 included some geometry.[1] Arithmetic was taught in the *Secunda*, and in the *Prima* (highest class) the elements of astronomy and a few theorems from Book I of Euclid.[2] No mention is made of geometry being studied at Zittau,[3] and at St. Afra in Saxony geometry was not on the program in 1602. In the Gymnasium at Zwickau, Saxony, opportunity was given as early as 1521 for the study of geometry, but it was optional. Students who wished could listen to lectures on arithmetic, geometry, and astronomy, the classes meeting on Saturdays. The regular course assigned "Rechnen" to the *Quinta* and astronomy to the *Tertia*.[4]

The study of the "sphæra" and arithmetic generally constituted the work in mathematics in the schools of the sixteenth century. This is shown in the school "Ordnungen" of Goldberg[5] (1546), Würtemberg[6] (higher cloister schools, 1582), Brandenburg[7] (1564), and Cologne[8] (1543). At the Augsburg[9] Gymnasium (1576) arithmetic was taught in the fifth class but not in the higher classes. Geometry was not taught in the regular work but some mathematics was given in public lectures. The "Ordnungen" of Kursaxony[10] (1528), and of the church schools of Brunswick[11] (1543), Wittenberg[12] (1533), Hanover[13] (1536), Schleswig-Holstein[14] (1542), Pomerania,[15] (1563), Brieg[16] (1581), and Lower Saxony[17] (1585) make no mention o any of the mathematical branches.

By the end of the sixteenth century the teaching of geometry was the exception in the secondary schools of Germany. There

[1] For thirty years after its foundation (1538) not even arithmetic was taught. Russell, *German Higher Schools*, p. 42.

[2] Friedrich, p. 27.

[3] *Ibid.*, p. 28.

[4] In Germany, the classes are numbered the reverse of the practice in the United States.

[5] Vormbaum, *Die evangelischen Schulordnungen des achtzehnten Jahrhunderts*, Vol. I, p. 54. Hereafter referred to as Vormbaum.

[6] *Ibid.*, pp. 110-111.

[7] *Ibid.*, p. 538.

[8] *Ibid.*, p. 409.

[9] *Ibid.*, p. 470.

[10] *Ibid.*, p. 1.

[11] *Ibid.*, pp. 8, 44.

[12] *Ibid.*, p. 27.

[13] *Ibid.*, p. 32.

[14] *Ibid.*, p. 34.

[15] *Ibid.*, p. 165.

[16] *Ibid.*, p. 297.

[17] *Ibid.*, p. 396.

seems to have been two causes for this. The universities were still teaching it, and with increasing success, if we are to judge by the interest taken in the editing of the "Elements." Secondly, there was no demand for it in these schools from the practical side, for, as Günther[1] points out, the Gymnasia were interested in furnishing functionaries for the state and pastors for the churches.

In the seventeenth century, as a result of the Thirty Years War, the educational institutions of all classes were nearly destroyed, and hence little progress could be expected in the teaching of geometry. The reforms of Ratke touched only a little on mathematics, but those of Comenius tended to unite the study of mathematics and natural science of his time. According to Günther, it is difficult to prove that any school was influenced in its mathematical program by this great teacher, but one can admit an indirect influence, in view of the fact that his "Orbis pictus" was admitted into the schools.[2] Regarding the character of the mathematical work of the schools in the latter part of this century, "the conception of academic study formed in the last quarter of the seventeenth century remained about 150 years as a model. The method and contents of the mathematical program in general remained the same. . . . In those days the young mathematician had a notion of heterogeneous matters—he needed general knowledge allied to mathematics, hence there was little intensity. This arrangement persisted into the first decade of the nineteenth century."[3]

At St. Afra in Saxony the mathematical program of 1602 was arithmetic for the first two classes, while the highest class studied the "sphæra" and the first rudiments of astronomy.[4] No geometry was taught, and, according to Friedrich, these same conditions existed up to the beginning of the eighteenth century. In 1605 the rudiments of geometry were taught at the Gymnasium of Coburg, and the subject was obligatory.[5] No text was

[1] *Op. cit.*, Ens. Math., p. 250.

[2] This book brought the pupil in touch with life's activities by means of pictures. Some of these touched upon applied geometry. See Compayré, *The History of Pedagogy*, tr. by W. H. Payne, p. 126.

[3] Günther, Ens. Math., pp. 252-253.

[4] *Ibid.*, p. 27. The "sphæra" was mathematical astronomy. The astronomy mentioned above must have been general astronomy.

[5] *Ibid*, pp. 251-254.

used and the work was practical. At the Pädagogium in the same place arithmetic was the only mathematical subject taught,[1] and at Kurfalz[2] (1615) the "sphæra" and geometry were taught only as electives. At the Gymnasium at Erfurt, arithmetic was the only mathematical subject taught up to 1615.[3] In the program extending from 1619 to 1624, mention is made of one period being given to mathematical instruction in the first class, but no mention is made of geometry.[4] In 1643, and also in 1647, the first and second classes combined studied the "sphæra" one period per week. Arithmetic was studied one period per week in all three classes, and geometry was taught although not especially mentioned.[5] At this time Schröter's geometry was used at Erfurt. It dealt with both plane and solid geometry, but was entirely practical, and included the rudiments of surveying.[6] Euclid was not taught at this time, for in 1666 the rector at Erfurt recommended the teaching of the "Elements." The teachers answered that it was too difficult.[7] The same advice was given in 1671, with the recommendation "that in mathematics the 'Elements' of Euclid together with 'ocular demonstration' be given one period weekly." The result was that the mathematics teacher, Gruvius, believing Euclid too hard, wrote in 1671 a geometry based on the "Elements." This was used in the school at Erfurt. The books of both Gruvius and Schröter contained many exercises of a practical nature.[8]

At Gotha[9] (1605, 1626) arithmetic and geometry were taught in the fifth of the six classes. In the Gymnasium at Jochimsthal[10] (1607), geometry was taught in the highest of the three classes. At the Giessen[11] Pädagogium (1605-1623), considerable

[1] Vormbaum, II, p. 24. Cf. Hellman, *Mathematischer Unterricht an den Erfurter evangelischen Schulen im* 16. *und* 17. *Jahrhundert*, II, pp. 4-5. Hereafter referred to as Hellman.

[2] Günther, Ens. Math., pp. 251-254.

[3] Hellman, p. 4.

[4] *Ibid.*, p. 5.

[5] *Ibid.*

[6] *Ibid.*, p. 7.

[7] *Ibid*, pp. 5-7.

[8] *Ibid*, p. 10.

Simon Stevin in his *Tomus secundus mathematicorum hypomnematum de geometriæ praxi* emphasized the practical side of geometry. Ed. 1605.

[9] Vormbaum, II, p. 30.

[10] *Ibid.*, pp. 72, 75.

[11] Monumenta Germaniæ Pædagogica, 28, p. 29.

attention seems to have been given to the study of mathematics, the work including the "sphæra," geometry, geodesy, and cosmography. Mathematics and geography were studied in the Gymnasia at Beyreuth[1] (1664) and at Liegnitz[2] (1673). The school "Ordnung" of the state of Brunswick[3] (1688) provided for the teaching of practical geometry in the Ritterakademie at Wolfenbüttel, the work, however, being done in private hours together with the arithmetic. In the Gymnasia at Gorlitz[4] (1609), Beuthen[5] (1614), Soest[6] (1618), Moers[7] (1635), Stralsund[8] (1643), Kronstadt[9] (1644, 1657), and Halle[10] (1661), the study of geometry was still neglected. The same was true at the Lyceum at Sorau[11] (1650) and in the Latin Schools at Emden[12] (1621) and at Frankfort-on-the-Main[13] (1654). The church schools of the state of Brunswick gave little attention to the study of mathematics. We do not find that geometry was on their programs before 1741.[14]

Thus the seventeenth century saw geometry not yet generally taught in the secondary schools. The influence of Euclid was just beginning to be felt in some of the schools. On the whole, practical geometry was more commonly taught than in the previous century. There was the usual practical work in connection with surveying. As for method, little can be said. In logical geometry we recall that the rector at Erfurt in 1671, recommended the teaching of Euclid with "ocular demonstrations." This would seem to indicate some sort of explanation from a diagram.

In the eighteenth century, we find mathematics and science becoming more prominent in the secondary work. This would be expected after the great development in these branches in the previous century, in which the names of Galileo and Kepler

[1] Vormbaum, II, p. 629.

[2] *Ibid.*, p. 649.

[3] Monumenta Germaniæ Pædagogica, 8, pp. 244, 266.

[4] Vormbaum, II, p. 98.

[5] *Ibid.*, p. 117.

[6] *Ibid.*, pp. 206-207.

[7] *Ibid.*, p. 273.

[8] *Ibid.*, pp. 379-382.

[9] *Ibid.*, p. 384.

[10] *Ibid.*, pp. 522-529.

[11] *Ibid.*, p. 396.

[12] *Ibid.*, p. 260.

[13] *Ibid.*, p. 438.

[14] Monumenta Germaniæ Pædagogica, 7, pp. 49-196.

were associated with physics and astronomy, Descartes with the invention of analytic geometry, and Newton and Leibnitz with that of the calculus.

Some of the schools were quite late in beginning the teaching of geometry. At the Gymnasium at Zittau, no geometry was taught before 1707. It seems that it became elective about this time.[1] Under the influence of Christian Pescheck, the instruction became more common, but it was still elective and was taught in private classes. It was not until 1726 that regular instruction was given in mathematics in the upper classes, but no mention is made that this included instruction in geometry.[2] As for the program of 1740 and 1760, the work in mathematics was enlarged, but no special mention is made of geometry. From 1783 to 1798, the work in geography was widened, but the mathematics was reduced. In the upper classes it was again made elective for private work.[3] In 1803 the standards were raised, and students had regular hours in which some practical geometry was taught (two periods), but in their regular leisure hours (7-10 or 6-9, and 1-3) mathematical science was studied. In the leisure hours on Saturdays and Sundays (from 11 to 12), scientific geometry was studied by the combined *Prima* and *Secunda*.[4] Thus we see that, at Zittau, it was not until the beginning of the nineteenth century that practical geometry was required, and at that time the more scientific[5] study of the subject was made elective in private classes.

In the Gymnasium at Erfurt[6] from 1713 to 1743, geometry was still largely taught in connection with geography. The geometry of Gruvius was taught in the second class, but the study was not likely beyond the elementary constructions.[7] In 1762 each of the three classes had from one to one and one-half hours of mathematics per week. In the first class, "geometry" and "stereometry" are mentioned.[8] In the Gymnasium at Görlitz (1770), the study of mathematics was optional in the

[1] Friedrich (p. 29) states that the rector in 1707 believed that geometry and astronomy should be taught only in private classes.

[2] Friedrich, p. 29.

[3] *Ibid.*, p. 33.

[4] *Ibid.*, p. 34.

[5] Interpreted as synonymous with geometry based on logic.

[6] Hellman, p. 11.

[7] *Ibid.*, p. 12.

[8] *Ibid.*, p. 15.

upper classes.[1] As at Zittau, those taking this work met for instruction in their leisure hours on Saturday and Sunday. In the Fürstenschulen about the same time, the first year and a half was given to "Rechnen," and the remaining year and a half to the beginnings of geometry and applied mathematics.[2] Geometry was prescribed in 1702 at the Pädagogium[3] at Halle where the "Elementa geometriæ" of Andrew Tacquet[4] was "explained" in connection with field practice.

The school "Ordnungen" of some of the German states and cities not mentioned above show an increased interest in the study of geometry. We learn that the study was prescribed in the Gymnasia of Baden[5] (1705), at the Dormstadt Pädagogium[6] in Hesse (1752), and in Baden-Durlach[7] (1767) where great efforts were made to train the teachers for the work. The "Ordnung" of the city of Brunswick shows that about 1745 geometry and arithmetic were first taught in private classes in the Gymnasia Katharineum and Martineum.[8] It was not until 1801, however, that geometry was a prescribed study in the latter institution. Even then the *Prima* class met but once a week for this work.

Since the mathematical books of Christian Wolf exerted a wide influence on teaching in the secondary schools, it would be well to get some idea of their nature. Wolf wrote two important series of mathematical works, the "Elementa matheseos universæ" and the "Auszug aus den Anfangsgründen aller mathematischen Wissenschaften." The latter has many features like the former, the larger work, but is far simpler and more practical. The author evidently intended it primarily for use

[1] Friedrich, p. 32.

[2] Heym, *Zur Geschichte des mathematischen und naturwissenschaftlichen Unterrichts an Gymnasien, insbesondere an der Thomasschule in Leipzig*, pp. 17-18. Hereafter referred to as Heym.

[3] Vormbaum, III, p. 91. According to Hellman (p. 12) geometry was optional in 1721

[4] Tacquet, *Elementa geometriæ planæ ac solidæ, quibus accedunt selecta ex Archimedes theoremata*, 9th ed., 1694. Books I-VI, XI-XII of Euclid are generally followed, but the author shows independence of Euclid especially in his definition of parallel lines and in his "parallel axiom."

[5] Monumenta Germaniæ Pædagogica, 24, p. 355.

[6] *Ibid.*, 27, p. 273.

[7] *Ibid.*, 24, pp. LXII–LXIII, 121.

[8] *Ibid.*, 7, p. 196ff.

in secondary schools.[1] Both series were used, however, in the universities of Germany. We learn two things in this connection: geometry was still taught in the universities, and secondly, as we shall see, it was not necessarily according to Euclid. Wolf's "Elementa matheseos universæ,"[2] which first appeared in 1714, consists of five volumes. The first includes arithmetic, geometry (plane and solid), plane trigonometry, and analysis (including algebra, analytics, and calculus). The remaining volumes treat principally of mechanics, optics, perspective, geography, astronomy, navigation, fortifications, and architecture. The geometry is divided into two parts, plane and solid. The plane geometry consists of six chapters. Chapter I is devoted to definitions and axioms. Chapter II, to properties of lines, employing many constructions. Chapter III, to parallels and triangles. Chapter IV, to circles. Chapter V, to regular figures; and Chapter VI, to mensuration of plane figures. The solid geometry comprises five chapters. The first is devoted to definitions; Chapter II treats of planes; Chapter III, of the construction of solids; Chapter IV, of the mensuration of solids; and Chapter V is on gauging. The general sequence of subject-matter in the plane geometry resembles Euclid, but it is far from being copied after the "Elements." The theory of proportion is omitted, all facts on proportion being referred to the corresponding chapter in the author's arithmetic, which explains proportion algebraically. The treatment of parallel lines is not Euclidean. The author defines parallels in terms of their equidistance. On this as a basis, Euclid's parallel axiom is avoided, and Wolf therefore commits himself to illogical proofs. The area of a rectangle is found without considering the incommensurable case. Arithmetical computations are frequently inserted and frequent reference is given to the applications of geometry to surveying. On the whole, the geometry of Wolf, although based on logic, shows an independence of Euclid. The sequence of propositions in many

[1] The *Anfangsgründe* was used in the institutions at Halle and in other Gymnasia and universities of Germany. Vormbaum, III, p. 246; Heym, pp. 8-9; Hellman, p. 13.

[2] It was printed in Latin and went through many editions. Reference is here made to the 9th, which appeared in 1732. This shows the great popularity of the work.

instances is totally unlike that in the "Elements," and the methods of proof are many times different. The practical nature of the text and the simplified treatment of the logic (in some cases fallacious) marks a further departure from the geometry of the Greeks. The fact that Wolf's books were used in both the universities and the secondary schools shows that the teaching of geometry in Germany was by no means dominated by Euclid.[1]

Besides the books of Wolf, the geometries of Sturm (1635-1703) and Kästner (1719-1800) show progress in aim and method. The "Mathesis juvenilis"[2] of J. C. Sturm was written with special reference to academic teaching. The part devoted to geometry is, like the texts of Wolf, largely of a practical nature. The author grades the work for the different Gymnasium classes and also gives directions to the teachers who may use the books. Kästner[3] first treats the logical and then applies the same in mensuration and simple surveying. Euclid's axiom of parallels is used and the fundamental theorems on parallels are proved rigidly, but so briefly that it is necessary to supply some of the steps. Incommensurables are also treated in a scientific manner. On the whole the author seeks to make his work logically sound and at the same time to make it practical. Like the "Mathesis juvenilis" of Sturm, the text was specially prepared for academic use.

By 1762 some practical geometry[4] began to be taught in the Realschulen, which had recently come into existence. Hellman[5] tells us that "pure practical geometry without proofs" was

[1] The names of Gesner and Ernesti are associated with the teaching of mathematics at the Thomasschule at Leipzig in the eighteenth century. The latter wrote a book, *Initia doctrinæ solidioris*, in classical Latin, 1736 (Paulsen says in 1755), containing arithmetic, geometry, physics, and astronomy. It went through five editions up to 1796. The geometry was based on Euclid and laid emphasis on formal discipline. See Starke, *Die Geschichte des mathematischen Unterrichts in den höhern Lehranstalten Sachsens*, p. 21.

[2] In two volumes. Edition of 1711, 1716 referred to here.

[3] *Anfangsgründe der Arithmetik, Geometrie, ebenen und sphärischen Trigonometrie, und Perspectiv* (4th ed., 1786).

[4] This included constructions with the rule and compasses.

[5] *Op. cit.*, p. 15.

taught in the schools at Erfurt. We thus see the working down of elementary geometry into the intermediate grades.[1]

By the end of the eighteenth century, then, geometry was generally taught in the secondary schools of Germany. It was not the pure logic of Euclid, but was based on logic. Freyer at Halle and Sturm at Altdorf were far from using Euclidean proofs.[2] The texts used show that the practical element was not ignored. The universities still taught the subject, but they did not adhere to the "Elements," judging from some of the texts used. Also, the Realschulen began to teach a little geometry of a practical nature. Some general tendencies in the teaching of mathematics during this century are mentioned by Günther.[3] Francke in the Pädagogium at Halle gave mathematics for the first time an equal footing with the other subjects.[4] Semler at Halle, Hecker at Berlin, and others dared to abandon somewhat important points in the old plan of study in placing the classical languages in the "rear" and bringing mathematics and natural science into positions of importance.

Concerning methods of teaching geometry, we observe in the eighteenth century some hints on schoolroom practice that are not at variance with methods employed in Germany to-day. Demonstrative work on the part of the pupil seems to have been more insisted on at Halle. From the "Schulordnung" for the Pädagogium at Halle in 1721, the recommendation is given that students be prepared in geometry so that they can demonstrate more easily.[5] We learn that it is directed that figures be drawn on the board, and that pupils copy them in their books. This shows that the blackboard was then used and that the practice of keeping note-books in geometry was in vogue. In the geometry taught in the Pädagogium at Halle in 1721, the teach-

[1] The Realschulen occupy a place intermediate between the common schools (Volksschulen) and the Gymnasia. They prepare students primarily for the life of the middle classes.

[2] Hellman, p. 12.

[3] Ens. Math., pp. 262-263.

[4] Günther refers to Beier, *Die Mathematik im Unterrichte der höhern Schulen von der Reformation bis zur Mitte des* 18. *Jahrhunderts*, im Bericht über die Realschule II. Ordnung zu Crimmitzschau auf das Schuljahr 1878-79, p. 21.

[5] Vormbaum, III, p. 246. Cf. Heym, pp. 8, 9.

ing aim was two-fold, that of sharpening the wit and of making the work practical.[1] The form of questioning suggested in the above mentioned "Ordnung" shows efforts to stimulate exact thinking. A line is drawn on the board and the following questions and answers given:

1. Was ist das? A. eine Linie.
2. Warum ist es eine Linie? A. weil es in die Länge gezogen ist.
3. Was ist denn nun eine Linie? A. was in die Länge weg gezogen ist.

(Dies ist das erste Merckmahl, woran man eine Linie von andern Sachen unterscheidet: aber noch undeutlich.)

4. So ist ja dieser lange Tisch auch eine Linie? A. nein.
5. Warum nicht? A. weil er breit und dick ist, dass ich viel Linien drauf und dran ziehen könte.
6. Was muss den bey einer Linie nicht seyn? A. keine Breite noch Dicke.
7. Was muss man aber da seyn? A. die Länge.
8. Was ist nun eine Linie? A. eine Länge ohne Breite und Dicke. (Das ist nun nichts anders, als die ordentliche Definition einer Linie: und zugleich auch der Weg, wodurch die mathematici zu solcher Definition kommen.)

We learn,[2] also, that in the Fürstenschulen at Halle the authorities recommended that the pupils be encouraged to learn their geometry understandingly and not by heart. These statements as to methods employed refer to the schools at Halle only, but as the work there was of a high order, it tells us perhaps the most that can be said regarding method.

The beginning of the nineteenth century marked the period of the reorganization of the Gymnasia of Prussia. Under the influence of Frederick Wolf and Humboldt, the first general course of study took effect in 1816. Two or three recitations per week for each class in mathematics had sufficed before this time. As six periods per week were now given to mathematics, it can be judged that this study began to rank on a more nearly equal footing with the classics. But in 1827 a reaction set in and the number of periods per week was reduced to four, and in 1837 to three periods for the *Quinta*, *Quarta*, and *Tertia*. In 1882 the total number of periods devoted to all the classes in mathematics in the Gymnasia was thirty-four. In the same year the total number of periods per week in the Realgymnasia

[1] Vormbaum, III, p. 247.

[2] *Ibid.*, pp. 631-632.

were reduced from forty-seven to forty-four, and in 1892 this was reduced to forty-two.[1] We thus see that the impetus given to the study of mathematics in 1816 did not last and this was because influences were brought to bear that made the classics predominant, and hence the time given to mathematics was correspondingly diminished.[2]

In 1788 the Prussian government passed a regulation requiring all candidates to pass an examination before entering the universities. This was revised in 1812, but it appears, however, that the universities still admitted certain persons by special permission who had not passed these examinations. In 1844 an ordinance was passed that the Gymnasia exclusively were to prepare students for these examinations.[3]

During the first three decades of the nineteenth century the teaching of mathematics in the Gymnasia of Germany was not very intensive. Thus trigonometry was not taught as a distinct subject in Bavaria before 1860.[4] It has already been mentioned that during the eighteenth century the practical side of geometry was still emphasized and that Euclid as such was not taught in the secondary schools. To-day, in Germany, geometry is taught with reference to the other mathematical branches; the texts are not Euclidean in the strict sense of the term. As there was no sudden change in the method of teaching geometry during the last century, it is safe to judge that this unifying process in the teaching of mathematics was gradually evolved from the methods laid down by Wolf and his followers in the eighteenth century.

To summarize, the first logical geometry to be taught in Germany was in the universities. Previous to this time, the church schools taught only a few definitions and some practical exercises. In the sixteenth century, the secondary schools gave some little attention to geometry, but the work on the whole seems to have been optional. In the seventeenth century it was more widely taught in the secondary schools, where it seems to have been largely practical. During this century texts began to appear

[1] Russell, *op. cit.*, p. 312.

[2] Pietzker, *L'enseignement mathématique en Allemagne pendant le XIX^{me} siècle*, Ens. Math., 1901, pp. 77-78.

[3] *Ibid*, pp. 78-79.

[4] Günther, Ens. Math., p. 263.

which were used in the secondary schools. In the eighteenth century, better books were written, combining the logical and practical features of the work. Such were the texts of Wolf, Sturm, and Kästner. The latter book was quite academic in character, and, as it was much used, we can see that the schools were paying more attention to the pedagogic character of the work. During this century, the Realgymnasia were established, and mathematics obtained in these schools a ranking along with the classics. At this time geometry began to influence the work of the elementary schools, as simple constructions and other practical work began to be taught in the Realschulen. More attention was now paid to demonstrative work and pupils were more and more cautioned not to learn their work by heart.

Two features in particular stand out in this development. Geometry, which began in the universities, gradually worked its way down into the lower schools. Instead of being regarded as a finishing study at the close of the eighteenth century, it was regarded as a necessary preliminary study for the higher work. The other feature to be mentioned is the practical character of geometry when it found a place in the secondary schools. The universities in the Middle Ages taught the applications of geometry, but separate from the logical work. In the early secondary schools, the geometry taught was in connection with its applications in astronomy and geography. In the eighteenth century, attention was still given to applications in surveying in connection with the regular class work. The tendency grew, however, to confine the teaching to the logic and the ordinary applications in mensuration. In the nineteenth century, as will be shown later, the tendency has been to intensify the study of the several mathematical branches, the practical in geometry being limited almost entirely to its own field.

FRANCE

It was not until the eighteenth century that science found any recognition in the secondary schools of France. Previous to this time literary education occupied almost exclusively the field, and what science did exist was taught in the higher classes as a branch of philosophy, just as history was associated with the humanities.[1] Mathematics was not included as a branch of

[1] Sicard, *Les études classiques avant la révolution*, pp. 188-189.

philosophy but occupied a separate field, although Rollin and others looked with disfavor on this isolation of the subject.[1]

In the sixteenth century there was practically no science or mathematics taught in the colleges, although Erasmus advocated a place for natural history, geography, physics, mathematics, and even history.[2] We recall that in 1536 the first six books of Euclid were required for the master's degree at the University of Paris. The statutes of 1598 (art. 40) show that in the Faculty of Arts the sciences and mathematics were passed over almost without mention. It was recommended that students study the sphere[3] at 6 A.M., "with the help of some books of Euclid."[4] The effect of this program, a heritage of the Middle Ages, was almost nil. We can thus see that if mathematics was so little studied at the university at the end of the sixteenth century, the colleges could have paid but little attention to that subject. The newly discovered sciences of the seventeenth and eighteenth centuries made a change in the attitude of the institutions, and by the middle of the latter century science and mathematics were generally recognized.

The practical side of geometry was studied fully as much as in Germany, if one is to judge from the texts printed. The interest of Orontius Fineus in the practical side of geometry has already been mentioned. Charles Bouvelles was another writer of the sixteenth century. His geometry was not practical in the same sense as that of Fineus, for it included neither mensuration nor surveying. Neither was it on the lines of Euclid, although it showed the Euclidean influence in the nature of the subject-matter. This work appeared first in Latin in 1503,[5] and was followed in 1542 by an edition in French.[6] As was customary in later texts

[1] Thus at the Collège Mazarin mathematics was a separate course.

[2] Sicard, *op. cit.*, p. 190; Douarche, *L'université de Paris et les Jesuites*, p. 154.

[3] This had reference to "sphæra," or mathematical astronomy.

[4] Sicard, *op. cit.*, p. 190. Cf. Hahn, *Das Unterrichts-Wesen in Frankreich mit einer Geschichte der Pariser Universität*, pp. 99-100.

[5] Cantor, II, p. 379.

[6] Bouvelles, *Livre singulier et utile, touchant l'art et pratique de géométrie*, Paris, 1542. His *Le livre de l'art et science de géométrie*, 1511, was the first geometry printed in French (see "Nouvelle Biog. Générale," Vol. 6, Paris, 1862).

the preface contains some remarks on the origin and history of geometry. Mention is made of Pythagoras, Archimedes, and Euclid. Geometry and arithmetic being compared, the author says arithmetic is the soul and geometry the body, "both have attributes in common, but geometry is dependent on arithmetic."[1] This shows how far Bouvelles was from the influence of the "Elements" of Euclid. Four chapters are given to plane geometry and two to solid. The book closes with two chapters on the relation of geometry to symmetry as seen in the animate and inanimate world. The author proves few theorems, some being mere statements of facts.[2] He sometimes generalizes from special cases without giving the necessary steps. Thus he states that in an isosceles right triangle, the acute angles are each one-half of a right angle, and hence the angle sum of the triangle is two right angles. He then concludes that the angle sum in any triangle is two right angles. In solving the problem, "To find a mean proportional between two given lines," Bouvelles says, by way of introduction, that the Germans are accustomed to drink and eat on square tables, and the French on tables longer on one side than the other. "It is proposed," he says, "to reduce a French table to a German table." There is a suggestion here, at least, of good pedagogy.[3] The book on the whole may be classed as one in which the logical aim is in evidence, but it is also clear that the author sought to touch the lives of the people.

The geometry of Peter Remus (1569),[4] while essentially logical, shows a marked departure from Euclid in the sequence of subject-matter and the treatment thereof. Parallels are treated as in Euclid but the theory of proportion is omitted.

In the beginning of the seventeenth century, Jean Errard[5] wrote his edition of Euclid, and in 1678, Nicholas Mercator

[1] Bouvelles, *op. cit.*, p. 4.

[2] Thus when showing how to draw two lines equidistant, he says that two lines having a common perpendicular will not meet, by virtue of this perpendicularity, fol. 8 (verso).

[3] Bouvelles' book was very popular, for it went through many editions. The Latin was printed in 1511. The first French edition was printed in 1542. This was followed with some alterations by editions in 1547, 1551, 1555, 1566, 1605, and 1608.

[4] Edition of 1627 referred to here.

Les neuf premiers livres des élémens d'Euclide, 1605.

produced a work[1] showing the influence of Euclid, but its independence of the "Elements" is quite marked. The author emphasizes the idea of motion, using this especially in his treatment of proportion. Parallel lines are defined as lines everywhere the same distance apart. Mercator attempts to prove the fundamental theorems on parallels without assuming the axiom of parallels or its equivalent, and hence the reasoning is fallacious. The logic is also faulty in other places.

The geometry of Arnauld, which appeared in 1667, also shows an independence of Euclid. The book is primarily a logical geometry and has for its aim the presentation of plane geometry in a logical manner, but "in a new order and with new demonstrations of the most common propositions."[2] Arnauld breathed a new spirit into elementary geometry and had great influence on the succeeding text-books.[3] Thus through these centuries we see independence of Euclid in France as well as in Germany.

The geometry of Le Clerc[4] (1669) is an interesting book from a pedagogical standpoint. As the author states, it represents a new and singular method. The left page of the open book is given up to the written work and the right page to pictures which are supposed to illustrate the theorem or problem under discussion. Le Clerc goes beyond the "Orbis pictus" of Comenius, which appeared eleven years earlier. Some of the representations are ridiculously fanciful. Thus, he is considering the problem, "Through a given point to draw a line parallel to a given line." The picture which corresponds to this is that of four men with swords, two of whom are fighting a duel. Two of the swords are parallel in position while the other two represent auxiliary lines. Le Clerc certainly carried his idea to the extreme, but his fundamental idea was correct, that of making the

[1] *In geometriam.*

[2] See preface, Arnauld, *Les nouveaux élémens de géométrie.* Ed. 1683. The text is fully described by Bopp in his *Antoine Arnauld, der Grosse Arnauld als Mathematiker*, in Abhandlungen zur Geschichte der Mathematischen Wissenschaften, 1902, pp. 189-336.

[3] Thus, see Lamy, *Les élémens de géométrie ou de la mesure de l'entendue*, 1710. In the preface, the author remarks that he does not follow Euclid's sequence, but, like M. Arnauld, follows the natural order.

[4] Le Clerc, *Pratique de la géométrie sur le papier et sur le terrain.*

work have an appearance of reality. The subject-matter of the book is concerned only with the construction of plane figures, no proofs being given.

Ozanam produced a work[1] in 1699 which is an exact reproduction of Le Clerc's book with the exception of some slight alterations of a few of the figures. Ozanam's book is in both French and German. No acknowledgment is made of the text of Le Clerc, but there can be no doubt but that it was simply an edition of the former work.[2]

Up to the time of the expulsion of the Jesuits (1762), the secondary education of France was largely under their control, and mathematics was generally neglected by them. The opinion of the Abbé Fleury in his "Traité du choix et de la méthode des études" (1686)[3] is indicative of the value set upon the teaching of mathematics. He chose as indispensable studies for all, religion and morals, civility, and logic and metaphysics. Grammar, arithmetic, economics, and jurisprudence were considered as studies of second degree; while among those of third degree were Latin, rhetoric, history, natural history, and geometry. The Jesuit College of Clairmont (now the Lycée Louis le Grand) certainly recognized in the early part of the seventeenth century the value of algebra and geometry, if one is to judge from a collection of texts or pamphlets[4] printed under the name of the college and written by some of the professors. The author of one of these pamphlets states that the order of Euclid should not be followed in using no theorem unless it has been proved. "The elements of geometry can be taught far more easily and briefly, and not less thoroughly, if some theorems are assumed, to be proved later when there is need." Here is a suggestion of good pedagogy that anticipates the "Perry Movement" of to-

[1] *Neue Übung der Feldmess-kunst. So wohl auff dem Papier als auff dem Feld.*

[2] For a more extended discussion concerning these related works, see Graf, *Die Geometrie von Le Clerc und Ozanam, ein interessantes mathematisches Plagiat aus dem Ende des XVII. Jahrhunderts.* In Abhandlungen zur Geschichte der mathematischen Wissenschaften, 1899, pp. 115-122.

[3] Lantoine, *Histoire de l'enseignement secondaire en France au XVII^me^ et au début du XVIII^me^ siècle*, pp. 188-192.

[4] Bound as one volume under the heading, Bussey, *Encyclopœdia mathematica collegii Claromontani Parisiensis*, societatis Jesu, 1638.

day.[1] The practice of printing texts or outlines of the mathematical subjects was not confined alone to the College of Clairmont. In 1689 a pamphlet of thirty pages was printed by Riviere in the Collège Mazarin under the title, "Theses mathematicæ de geometria elementari tam speculativa quam practica." Theorems are merely stated. The practical features of the work are shown in the problems, constructions, and the mapping of figures from the field.

The influence of the Port-Royalists in perfecting the methods of mathematics and science should not be passed unnoticed.[2] Arnauld, whose epoch-making geometry we have considered above, was a prominent member of this society, as was also the great Pascal.

After the first quarter of the eighteenth century, the teaching of mathematics grew more in favor from year to year in the colleges. It was not until 1789, however, that mathematics began to flourish there.[3] By 1730 text-books in geometry began to be common in the colleges. Previous to this time, dictation by the professors was the common practice.[4] That this method was in use at the beginning of the eighteenth century is shown in the preface of Rivard's text,[5] where the author remarks that his book is compiled from notes on the lectures delivered by his professor of philosophy. "They began to suppress dictation made by professors of philosophy in the class, and put in the hands of pupils elementary books, written in French, which different authors began to prepare on all sides."[6] Sicard mentions the books of Rivard, Clairaut, and La Caille as being used in the class work.[7] While the colleges by 1730 were using books printed in French, the university did not pursue a similar course until 1789.[8] As the book of Rivard was

[1] The "Perry Movement" will be considered in Chapter VII.

[2] Cadet, *Port-Royal Education*, trans. Jones.

[3] Sicard, *op. cit.*, p. 352. [4] *Ibid.*, pp. 204, 351.

[5] Rivard, *Élémens de mathématiques*, 1744 (4th ed.).

[6] Sicard, *op. cit.*, p. 204.

[7] The Abbé Leroy in his "*Lettre sur l'education publique*," Brussels, 1777, p. 241, also mentions the use of Rivard's book in the colleges in 1730.

[8] The students petitioned for French texts in the different branches of philosophy, so the university asked its professors to write elementary books for the different courses. The first written was one on physics. Sicard, p. 358

used in the colleges (1730) before the publication of the other works above mentioned, it is stated with some certainty that this book was the first elementary geometry in the French language to have any general use in the schools of France.

We learn one other important fact from the preface in Rivard's book. The author states that his book is compiled from notes taken in the class of philosophy at the university. This shows us the nature of the teaching at the University of Paris at the beginning of the eighteenth century. In geometry it corresponded to what is given now in the average American high school. The nature of Rivard's book gives us the right to make this assertion. The "Élémens de mathématiques" comprises in a single volume arithmetic, algebra, geometry, and trigonometry. The geometry is divided into three parts, the geometry of lines, of surfaces, and of solids. The author does not treat proportion in his geometry, but refers to the chapter devoted to that subject in his algebra. The subject of incommensurables receives little attention. Parallel lines are defined in terms of their equidistance, and the fundamental theorems on parallels are not rigorously proved. Thus Rivard says[1] in one of his proofs, "the corresponding angles are equal, as one can see." On the whole, the book is written in a scholarly fashion, notwithstanding some logical *lacunes* as just shown. The aim of the book is evidently to improve the student in logical thinking. The practical aim is not in evidence, for there are no applications to surveying, and mensuration is not prominent.

The geometry of La Caille[2] (1741) has the same disregard for logical rigor in the treatment of parallels. In the introduction it is stated that this book was prepared for use in the Collège Mazarin and that four hours per week would be given to elementary mathematics.[3] Clairaut also in 1741 published his "Élémens de geometrie." It is even less rigorous in its logic than the geometry of Rivard. The subject of parallels is treated in the same loose fashion. Constructions are frequently given

[1] *Op. cit.*, p. 24.

[2] *Leçons élémentaires de mathématiques.*

[3] Ces leçons seront expliquées au Collège Mazarin, tous les ans depuis la Saint Martin jusqu'au Carême, les Lundis, Mardis, Jeudis, & Vendredis, à une heure & un quart après-midy."

without proofs in any strict sense.[1] Clairaut published another edition in 1765. It has the features just mentioned, but here the author boldly attacks the rigor of Euclid. In the preface he says, "It was necessary that Euclid should prove that intersecting circles have not the same center, because he had to contend with the obstinacy of the Sophists. It was necessary to have this geometry as a *logic*, but to-day times have changed." The author says he neglects many propositions because they are of no use in themselves. Our reformers to-day could not be more radical in this respect. In other ways less striking, Clairaut shows pedagogic insight. He does not begin his geometry with the customary definitions and principles, but places them where needed in the body of the text. In the preface the author calls attention to this feature. From the standpoint of aim, Clairaut states that his book has the double object, the measuring of land, and the discovery of geometric principles.[2] On the whole the book is noteworthy in that the author recognized that a book logically perfect is not necessarily the best book for use in the schoolroom. He was ready to sacrifice logic for the sake of interest and practical necessity.

In the last half of the eighteenth century, as the military schools increased the efficiency of their work, mathematics received more attention, and texts written for use in these schools began to appear. One of the first of these was the "Cours de mathématiques" of Étienne Bézout, which includes mechanics, arithmetic, geometry, algebra, and astronomy. The geometry includes plane and spherical trigonometry as separate subjects. Like those of Rivard and Clairaut, the geometry is divided into three parts, the geometry of lines, of surfaces, and of solids. It is interesting to note that the geometry of planes in space is treated under the heading of surfaces. This is not the usual arrangement, for solid geometry generally includes the treatment of planes in space. Bossut[3] in 1789 wrote a text on mathematics with a particular object in view. In the preface the author states that the mission of his book is to help make uniform the teaching of mathematics in the different royal military schools. It is based on the course given at the École Royale

[1] Thus from a point without a line it is required to draw a perpendicular to that line. The author solves this by basing it on the case where the point is on the given line. But the proof is lacking in the latter case.

[2] p. xiij.

[3] *Cours de mathématiques.*

Militaire at Paris.[1] A review of the book will give us an idea of the nature of the geometry taught there at that time. As was customary in France and Germany, the various branches of elementary mathematics were embraced in one book. This contained arithmetic, algebra, geometry, and trigonometry. Chapters I–VII are on plane geometry. Chapter VIII considers the "properties of planes," and Chapters IX–XII are on the geometry of solids. Chapter XIII contains the elements of plane and spherical trigonometry. This is followed by a chapter on the "applications of algebra to geometry," in which the conic sections are treated.[2] The book has the general characteristics of the books of Clairaut and Bézout, that is, it emphasizes the practical phase of the subject; but not so much as one would expect since it was especially prepared for the military schools.[3] Its logic is faulty in places, in particular in the treatment of parallel lines.

Legendre's "Éléments de géométrie," which appeared in 1794, obtained a wide-spread popularity both in Europe and in the United States. Its general recognition was due to two causes. It abandoned the sequence of Euclid and so simplified the subject-matter. Secondly, it was logically sound, and hence was recognized by the mathematical world. Legendre departed from Euclid in two important respects. First, his sequence differs from that in Euclid. The sequence in the first six books of Euclid is as follows: Book I is on the geometry of lines; Book II, on areas; Book III, on circles; Book IV, on regular figures; Book V treats the theory of proportion; and Book VI applies this to plane figures. Legendre's sequence in plane geometry is: Book I, on lines; Book II, on circles; Book III, on proportion applied to plane figures; and Book IV, on regular polygons and the mensuration of the circle. Book III in Legendre includes parts of Book I and Book VI of Euclid. Legendre does not treat the theory of proportion, but refers the reader to the treatment of that subject in arithmetic and algebra.

[1] In the preface, the author also says that his book may be a little "extreme" for those pupils not intending to be military engineers; e. g. for those preparing for the infantry or cavalry. He says the "learned professors know how to abridge the work in such cases."

[2] One finds here the use of *aa* for a^2, which is a relic of the old algebraic symbolism.

[3] It was also used in the Collège of Sorize, as was also the geometry of Bézout. Leroy, *op. cit.*, p. 132.

This reference to arithmetic and algebra points out a second important difference between the two geometries. Euclid treats geometry within its own domain. He not only does not base any part of his geometryon arithmetic, but does not illustrate its applications in that field. Legendre assumes the correspondence between a line segment and number. He applies the algebraic method to establish theorems and gives attention to mensuration. In one other respect did Legendre depart from Euclid, in admitting hypothetical constructions. Euclid permitted the demonstration of no theorem unless the necessary constructions had already been made. In disregarding this restriction, Legendre followed the example of some of his less illustrious predecessors. We cannot claim for a moment that Legendre was the first to abandon Euclid in the sequence or treatment of the subject-matter of elementary geometry, nor was he the first to abandon Euclid's order and still maintain a sound logical structure. In our treatment thus far, several types of geometries have been mentioned. There were the various editions of Euclid, printed in the different countries. Also we have examined the practical geometries, which have had little to do with logic. Then there were combinations of these two extremes. That is, there were geometries embracing a large part of the subject-matter o Euclid (but not in its sequence) which tended toward the practical. In these, logical rigor was sometimes sacrificed. Such were the geometries printed in France in the eighteenth century and before Legendre. As we have seen, the same class of books appeared in Germany. Instead of thinking of Legendre as rescuing the mathematical world from the tedious servitude to Euclid, we are rather to think of him as placing geometry again on a sound logical basis, but differing from Euclid in order and method. The book of Kästner was sound logically and departed from Euclid's sequence, but its virtues did not equal those in the "Éléments" of Legendre, and hence we credit Legendre with making the first logical departure from Euclid that the world recognized.[1]

[1] We have seen that many of the texts before Legendre failed in a logical treatment of parallels in not using the parallel axiom or its equivalent. In his several attempts in different editions of his work to avoid using this axiom, Legendre also fails in his logic, skillful though he was. Legendre recognized that Euclid assumed a great deal in his parallel axiom and hence sought to prove its equivalent. Although he failed, we must, however, consider the work as a whole highly logical. See Legendre, *Éléments de géométrie*, editions 1794, An VIII, An IX, 1812, and 1832.

In the eighteenth century geometry was taught in France with increasing interest, owing in part to the importanceof this subject in the military schools, and also undoubtedly because the Jesuits, who had not favored the mathematical studies, had been expelled from France in 1762. During the years from 1794 to 1808 the developing tendencies in mathematics received a check, for the university and most of the colleges were closed.[1] The founding of the École Polytechnique and the École Centrale at the beginning of this period tended, however, to keep alive this interest.

When Napoleon became emperor in 1808, he organized the imperial university and ordered the creation of lycées equal in number to that of the courts of appeal.[2] Many of the former colleges now received the title *lycée.* This institution corresponds to the German Gymnasium, or, to the American high school if we add to the latter about one year of college work and prefix to it the last two years of the grammar school. The lycées have emphasized the teaching of mathematics in different measure. In particular the Lycée St. Louis,[3] created Oct. 1, 1820, has had for its principal function the preparation of students for the different government schools, including the École Normale Superieure and the various military schools. Hence mathematics has always been especially emphasized in this particular lycée.

As was the case in Germany, the first half of the nineteenth century shows a struggle for supremacy between the literary and scientific studies, the latter including mathematics. From 1802 to 1852 there was a continuous struggle between the two in the lycées.[4] The result of this conflict was a separation of these courses. A recommendation to this same effect was made in 1802, but it seems not to have influenced the instruction in the lycées.[5] After the middle of the last century

[1] Chauvin, *Histoire des lycées et collèges de Paris*, pp. 43-45.

[2] *Ibid.*, p. 45. The decree that created the lycées was made May 1, 1802.

[3] Founded as the Collège d'Harcourt in Paris.

[4] Bersot, *Lettres sur l'enseignement. Histoire des plan d'études*, pp. 1-9.

[5] Although the lycées were established under a decree of 1802, they did not perform their functions until 1808. In 1802, a plan of study was tried in a special class of schools called the *Prytanée.* Bersot, p. 3.

the feeling grew that all students should not be subjected to the same program. We learn of "special courses" being given in some of the lycées. In 1890 a so-called "modern course" was added to the school program,[1] but the schools were nevertheless still dominated by the classical course. The new plan of study of 1902[2] marks an epoch in giving the student great freedom of choice in his selection of a program. Liberal provision has been made for the work in mathematics and science.

Since the time of Legendre, the French schools have in the main either followed his "Éléments" or used texts based on it. A text that rivalled Legendre's about the beginning of the last century was the geometry of Lacroix.[3] These, says Delambre,[4] were the two important texts in use between the years 1789 and 1810. Referring to the features of these books, Delambre comments on the efforts of Legendre "to treat some parts wholly analytically" and says that the book of Lacroix "can serve as an introduction to modern analysis."

That Lacroix was a worthy rival of Legendre in the field of elementary geometry can be seen in his essays on teaching. These show us what was engaging the attention of a teacher and mathematician at that time (1805). Referring to the correspondence between plane and solid geometry, Lacroix says,[5] "Many lines and figures traced on a plane are only particular cases of lines, planes, and solids considered in space; and it is indispensable, whenever possible, to teach in close relation those parts of plane and solid geometry that admit of analogous treatment." Thus we see a suggestion to teach parts of plane and solid geometry simultaneously.[6] The author lays emphasis

[1] Compayré, *The Reform in Secondary Education in France* (trans. by Finnegan), Educational Review, 1903, p. 134.

[2] *Plan d'études et programmes d'enseignement dans les lycées et collèges de garçons* (Arrêtés du Mai, 1902).

[3] *Élémens de géométrie* (á l'usage de l'ecole centrale des quatres-nations). 7th ed. (1808).

[4] *Rapport historique sur les progrès des sciences mathématiques depuis* 1789, Paris, 1810, p. 3.

[5] Lacroix, *Essais sur l'enseignement en général et sur celui des mathématiques en particulier*, 1838 (1st ed., 1805), p. 297.

[6] The author cites the trouble of finding a correspondence between the congruent triangles of a parallelogram and the non-congruent pyramids of a triangular prism (p. 297).

It is remarked that pupils at the age of fifteen or sixteen were studying solids (p. 299).

on analogy. He gives a suggested sequence of subject-matter as follows:

1. Straight lines with regard to length but not to situation.
2. Combinations of lines, congruent[1] and similar triangles.
3. Polygons, congruent or similar.
4. Combinations of lines and circles.

Lacroix then goes on to say, that from the above order, in the first part of the geometry, he builds up by analogy the rest of his work, relative to the measure of areas, to planes, and to solids. The author is keeping in mind the analogies between plane and solid geometry, for, he says, when Euclid I, 47, is placed in the early part of geometry, as in the "Elements," there is no analogous theorem in solid geometry, but when proved under similar triangles (not pure geometry, he says) there is an analogous theorem, which is, "The square of the area of the largest face of a tetrahedron, which has three contiguous faces mutually perpendicular, is equal to the sum of the squares of the areas of the other three faces."

Lacroix says experience teaches him not to separate the teaching of theorems and problems, and that in the construction work great care should be used to have the drawings exact. He also believes that simple surveying should be taught in this connection.[2] The author raises the question whether algebra should precede or follow geometry. Lacroix does not give an absolute answer, but says it depends upon the minds of the pupils. He says geometry, above all other mathematics, should be learned first, provided it is presented with respect to its applications, "either on paper or in the field."[3]

These reflections of Lacroix have a significance to us from the standpoint of present teaching. The practice of pointing out the analogies between plane and solid geometry is commendable. Lacroix even suggests that in some cases the proofs in the two geometries could be given in connection with each other.[4]

[1] The author uses the word "égal."

[2] Lacroix, *op. cit.*, p. 303.

[3] As regards style of demonstration, Lacroix says that the "reductio ad absurdum" should be used as little as possible for the reason that the best proof is that which follows in a chain of proofs (p. 307).

[4] This is being agitated in Italy and France at the present time. See below, pp. 109-112; 115-116.

We get some idea of the teaching in the colleges in 1843, as well as some suggestions as to method, from a report of M. Busset.[1] He suggests for a first reform that arithmetic and algebra be applied from the first in the teaching of geometry. As for the aim in teaching of geometry, Busset says it should not be that of developing the science of men, but that of arriving at a knowledge of nature.[2] The rest of the report is concerned with recommendations as to the treatment of specific portions of the subject-matter of geometry, arithmetic, and algebra. It is recommended in the teaching of geometry to employ the principle of duality and to make use of the method of analysis.[3]

To summarize, we find that geometry was beginning to be seriously studied in the secondary schools of France by the early part of the eighteenth century, and by the last quarter of that century mathematics as a whole was flourishing in the colleges. During the period from 1794 to 1808, when the university and the colleges were suppressed, the interest in mathematics was kept alive in the military schools founded by the government. During the last century these schools and the other government technical schools, by their rigid entrance requirements, have stimulated mathematical study in the secondary schools, and to-day the work in the lycées and colleges is shaped in a large measure by these demands.

The teaching of geometry in the secondary schools of the eighteenth century was not according to the strict logic of Euclid, if we are to judge from the subject-matter of the books which were used at that time. In these the logic was sometimes insecure. Legendre secured a proper balance in his "Éléments." His work was logical but at the same time was usable in schoolroom practice. The influence of Legendre has endured to this day, but the later French geometers did not confine themselves to the plan of Legendre. We have seen that early in the last century recommendations were given by Lacroix that showed an independence of method.

Concerning the method of study, it can only be said that up to the early part of the eighteenth century, learning by heart was the practice. Recommendations were made at that time that

[1] *De l'enseignement des mathématiques dans les collèges*, p. II, ff.

[2] Busset, *op. cit.*, p. xi.

[3] *Ibid.*, p. 28.

this be stopped and that the learning be rational. It is not to be judged that this evil was entirely corrected.

ENGLAND

About 150 years elapsed after Adelard of Bath translated Euclid from the Arabic into Latin before the "Elements" began to be taught at Oxford University. Roger Bacon (1212–1294), who lectured there and at Paris, recognized the importance of geometry and stimulated mathematical interest at both of these universities. Writing at the end of the thirteenth century, he says that at Oxford, few, if any, residents read more than the definitions and the enunciations of the first five propositions of Euclid.[1] By the middle of the fifteenth century a little greater interest was taken in this study, for we learn that at the same university, from 1449 to 1463, the first two books were read.[2]

Added interest was given to the study of Euclid over a hundred years later (1570), when Sir Henry Billingsley translated the "Elements" from the Greek into English.[3] Previous to this time there had been no professorship of mathematics at Oxford or Cambridge. About 1570 Sir Henry Savile began to give unpaid lectures on the Greek geometers at Oxford, and in 1619 the Savilian professor was Briggs, a Cambridge man, who began lecturing on Euclid I. 9, where Savile had left off.[4] Cambridge followed the example of Oxford, and in 1663 the Lowndean professorship of mathematics was founded in that university. About the same time Isaac Barrow, the teacher of Newton, made a complete edition of Euclid,[5] having published in 1660 an English translation for two of his pupils at Trinity College, Cambridge. This remained a standard for about fifty years.[6]

In 1702, William Whiston edited Tacquet's[7] Euclid. This remained a standard until the appearance of Simson's book.

[1] Ball, *A History of the Study of Mathematics at Cambridge*, p. 3.

[2] *Ibid.*, p. 9; Gow, p. 207.

[3] Ball, *op. cit.*, pp. 22-23.

[4] Gow, p. 208. The problem is "To bisect a given angle."

[5] In 1665.

[6] Ball, *op. cit.*, p. 46.

[7] The Euclid of Tacquet was printed in Antwerp. It was popular on the continent. See p. 67 above

About 1730 the usual texts of Euclid were the editions of Barrow, Gregory, or Whiston.[1] The next edition of Euclid to have wide-spread popularity was that of Robert Simson, which appeared in 1756. The editions of Euclid, following Simson, were more or less based on this book.[2] The texts of Playfair (1795) and Todhunter (1862) obtained great popularity in England and also in America. We thus see a wide-spread interest in Euclid from the standpoint of the text-books. There was no writing on the practical side of geometry such as prevailed on the continent up to about the middle of the seventeenth century.[3] Nor do we find texts combining the practical with the logical. Euclid reigned supreme.

The period from 1660 to 1730 marked the time when the study of Greek geometry was at its height in England.[4] The universities were now giving their attention to the new turn mathematics had taken after the invention of the differential calculus by Newton. With this new material to work with and to give their time to, we should expect that Euclid would receive some attention in the secondary schools. Just when this transition began it is hard to say. During the eighteenth century the average age of freshmen at the universities was gradually increasing,[5] and when boys stay longer at school, they necessarily begin to learn higher subjects. Hence there is strong probability that during this century Euclid was gradually being studied in the schools, and it may be safely guessed that its place among the school books dates only from the middle of the last century at the earliest.[6] If geometry was studied in the schools by the middle of the eighteenth century, the attention given to it must have been very slight. The great "Public Schools" were certainly very tardy in admitting Euclid into their course of study. We learn that Dr. George Butler, head master at

[1] Ball, *op. cit.*, pp. 92-93.

[2] Gow, p. 208.

[3] One MS. of the fourteenth century treated on the surveying of heights and distances. See Halliwell, *Rara mathematica*, pp. 56-71, where the work appears.

[4] Gow, p. 208.

[5] *Ibid.*, (note 2).

[6] Gow remarks (p. 208, note 2) that he can find no useful information on the curriculum of a public school before 1750.

Harrow from 1805 to 1829, introduced a little Euclid "lightly glanced at by the Sixth Form once a week." In the program of 1829, the Sixth Form (the highest) studied Euclid and vulgar fractions one period a week,[1] but geometry was not required at Harrow before 1837.[2] At the Edinburgh Academy (1835-36) the fifth class studied Euclid I; the sixth class, Books I–IV, when algebra was begun; the seventh class, the six books of Euclid completed. In the highest class, the seventh, trigonometry and algebra were both studied.[3] By 1839 at Rugby,[4] we find the Fourth Form studying Euclid I, propositions 1–15, and algebra being begun. By the close of the Sixth Form, Euclid VI was studied. At the Edinburgh Institution[5] six books of Euclid and the appendix of Playfair were studied. At this same time we learn something of the work and aims of the St. Domingo-House School.[6] This school prepared students in three lines: for the universities, for the army, navy, and engineering, and for trade. From the years of eight to ten the pupil gained ideas of the geometric figures; from ten to twelve he studied plane figures; from twelve to fourteen the geometry of solids together with trigonometry and algebra. From the years of fourteen to sixteen (the fifth class), spherical trigonometry, land surveying, navigation, and mechanics were studied. Those who intended to enter the universities studied in the sixth class Euclid I–VI in Latin.[7] Eton College admitted geometry into the course in 1836, but it was not required until 1851.[8]

In brief, the teaching of geometry in England has been of the extreme Euclidean type. The books in use have been the editions of Euclid, which excluded even the practical work in mensuration,

[1] Williams, *Harrow*, p. 85 and appendix c.

[2] Staunton, *The Great Schools of England*, p. 27. Also see *Quarterly Journal of Education*, Vol. III, p. 4, and Bache, *Report on Education in Europe*, p. 396ff.

[3] Bache, *op. cit.*, p. 368ff.

[4] *Ibid.*, p. 396ff (ref. *Journal of Education*, London, Vol. VII, p. 235ff).

[5] *Ibid.*, p. 388

[6] *Ibid.*, p. 402ff.

[7] The early age at which pupils studied geometry at this school argues that it must have been taught there for some years.

[8] Sharpless, *English Education*, p. 14; Staunton, *op. cit.*, p. 27

texts like those of Wolf in Germany, or Clairaut in France, not being in favor. As we shall see later, the teaching to-day closely follows these traditional lines. The universities were the first to teach Euclid, beginning as early as the thirteenth century. Toward the middle of the eighteenth century perhaps a little geometry was taught in the schools, but it was not until about the middle of the nineteenth century that the study of Euclid became common in the secondary schools of England.

RUSSIA

Mathematical works under the form of practical arithmetic and "the primitive art of measuring lines, areas, and volumes" have existed in Russia for a long time.[1] In the latter half of the sixteenth century, the government undertook the task of surveying the empire. Those who carried out this practical work possessed special knowledge drawn from various manuscripts on surveying. When this survey was undertaken it showed the necessity for mathematical instruction in the schools, but the government refused to further any plan to bring this about. Medieval instruction must have characterized the work in the schools as late as 1660, judging from a book used at that time. This was a sort of encyclopædia which embraced the seven liberal arts of the Middle Ages. The geometry included was essentially surveying.

The teaching of mathematics was neglected until Peter the Great saw the need of re-organizing the army and navy on the model of that of western Europe.[2] So at the beginning of the eighteenth century there were created many special schools where the teaching of mathematics took a prominent part. In these special schools the teachers either dictated the lessons

[1] Bobynin, *L'enseignement mathématique en Russie. Aperçu historique.* In Ens. Math., 1899, pp. 77-100; Hippeau, *L'instruction publique en Russie*, pp. xx-92; Beer und Hochegger, *Die Fortschritte des Unterrichtswesens in den Culturstaaten Europas*, vol. 2, pp. 5-105; Also see the article on Russia in Baumeister, *Die Einrichtung und Verwaltung des höhern Schulwesens in den Kulturländern von Europa und in Nord Amerika*, I_2, pp. 561-576. Where authority is not cited below, Bobynin is the scource consulted.

[2] Hippeau, *op. cit.*, p. xx; Beer und Hochegger, *op. cit.*, p. 5; Also see Bobynin.

to the pupils or read to them from their own note books, or from the books accepted as manuals. In the main the training was that of working by rule and applying what had been learned by heart.

In 1725 the Gymnasia[1] of Russia were created. Before this time geometry was taught (as were the other branches of mathematics) with reference to its utility, largely in connection with the engineering of warfare. The aim of the Gymnasium was not only to prepare for the practical professions, but also to develop pure science.[2] Geometry was taught in the two higher classes.

For thirteen years after the founding of these Gymnasia, mathematical teaching was done without text-books. The first geometry to be used as a text was written by Krufft[3] (inspector of Gymnasia) in 1732. Its title was "Kurtze Einleitung zur theoretischer Geometrie zum Gebrauch der studirenden Jugend in dem Gymnasia bey der Academie der Wissenschaften in Saint Petersburg," showing that the book was intended for academic use. The attempt of the Gymnasia to develop mathematics from a scientific point of view did not prove popular at first, for the attendance steadily declined for several years.

Another development in mathematical teaching in Russia was occasioned by the ecclesiastical schools taking up this study in 1743. The study was optional, however, even arithmetic being elective.

During this period most of the books on mathematics held to the dogmatic method. Such was the "Géométrie pratique" (1760) of Étienne Nasarof, which contained only definitions and problems with rules and solutions. A notable exception was the first algebra in the Russian language by Nicolas Mouravief (1752), in which book an attempt was made to replace the ancient dogmatic method by the demonstrative. Bobynin says

[1] The first teachers came from Germany. In the special schools just mentioned, some of the teachers came from England.

Beer und Hochegger (*op. cit.*, p. 50) give 1747 as the date of the founding of the first Gymnasium in St. Petersburg. Baumeister (*op. cit.*, p. 561) gives 1725, the same as given by Bobynin.

[2] The ecclesiastical schools emphasized the classical side of education.

[3] Krufft also wrote a text on mathematical geography. Both of these were in German, but were later translated into Russian

the schools preferred the dogmatic books, but from 1765, when Anitschkof[1] published his series of mathematics, the demonstrative method passed definitely into the Russian works on elementary mathematics.

About this time (1759), the Gymnasia first sent pupils to the university. Before this the students had come exclusively from the seminaries and ecclesiastical academies.

By 1786 the public schools had been organized. The parish schools prepared pupils for the district schools, those of the district for the Gymnasia, and the Gymnasia for the university. In the so-called principal public schools, which from 1781 to 1786 had not yet been consolidated with the Gymnasia, geometry was taught in the highest class.[2] The text-book[3] used represented the transitory period between the dogmatic and the demonstrative method.

At this time (1786), a special school, the Institute of Pedagogy, was created for the preparation of teachers for the public schools.[4] The Gymnasia later took over this function, the Institute preparing only for the Gymnasia. Each school, including the university, had as its duty the preparation of teachers for the inferior schools which preceded it in the school system.

The program of the Gymnasia at St. Petersburg from 1811 to 1816 embraced in the first class, arithmetic and algebra; in the second class, geometry and plane trigonometry; in the third class, applications of algebra to geometry, and the conic sections. In the fourth or highest class was included the applications of mathematics to physics.

To recapitulate, as late as the middle of the seventeenth century geometry was taught in Russia after the method of medieval instruction. It was embraced in the quadrivium and was entirely practical in its nature. By the beginning of the eighteenth

[1] One of these was on geometry, theoretical and practical, on the lines of Wolf in Germany.

[2] Beer und Hochegger, *op. cit.*, p. 105. Also see Bobynin.

[3] Bobynin gives its title, *Le petit manuel de géométrie.* It is divided into three sections: The measure of lengths, the measure of areas, and the measure of solids. Sixteen theorems are demonstrated. The rest is composed of definitions and problems, for the most part practical and dogmatically explained.

[4] Hippeau, *op. cit.*, p. 26. Also see Bobynin.

century the practical needs of warfare stimulated an interest in geometry and it was taught then in special schools with other branches of mathematics. It found a place in the newly founded Gymnasia in 1725, where it was valued more for its scientific character. Text-books in geometry began now to be used, the first appearing in 1732. The ecclesiastical schools took up the study in 1743, but as an elective. After 1786, the Gymnasia increased their functions by preparing teachers for the lower schools, and the teaching, which had formerly been characterized by memory work and learning by rule, began to lose its dogmatic character and appealed more to reason. It seems plausible that this change was finally accomplished by means of the new aim given to the teaching in the Gymnasia, that of the preparation of teachers.

HOLLAND

The seventeenth century, which was the most brilliant in the history of Holland,[1] was also the epoch which produced its great geometers.[2] During the seventeenth and eighteenth centuries about thirty books on geometry, forty on arithmetic, and twenty on applied mathematics were published.

The teaching of geometry in Holland began with the practical. The relation of geometry to drawing and perspective, and its applications to engineering, surveying, and the marine, were seen and appreciated. The Academy for Engineers was founded in 1660, and practical geometry was taught there from the beginning, the theoretical part being considered of minor importance.

Some of the treatises of the seventeenth and eighteenth centuries show the same practical trend. Ozanam in 1698 published at Amsterdam a complete course of mathematics. It was divided into five parts as follows: I, Introduction to mathematics, and the "Elements" of Euclid; II, Arithmetic, trigonometry, and sine tables; III, Geometry and fortifications; IV, Mechanics and perspective; V, Geography and gnomonics (dialling). Here we see attention given to Euclid, but the aim of the series is toward the practical. The work of De Graaf

[1] Cardinaal, *L'enseignement mathématique en Hollande.* In Ens. Math., 1900, pp. 317-339.

[2] Such men as Jean de Witt, C. Huygens, F. van Schooten, and J. Hudde.

(1694) treats essentially the same subjects, but the "Elements" as such is not included. These treatises are on the same lines as those followed later by Wolf in Germany. During this period works on geometry alone appeared, and usually under the name of the "Elements" of Euclid.

There is evidence that there was some independence of Euclid in the last half of the eighteenth century, for one geometry published announced a departure from the Euclidean method. This was the text of J. H. van Swinden (1746–1823) of Amsterdam. The texts of J. de Gelder (1765–1840) show a rigidity of method that indicates interest in the logical side. He wrote two geometries, the "Principles of Geometry" and the "First Principles of Geometry" (1827). The more advanced of his geometries was intended, as the preface indicates, for use in the university and for those interested in mathematics. The second was for use in the Latin schools.[1] In the preface of the latter work it is stated that the decree of 1815 required from future students of the university some knowledge of mathematics. We thus see a demand for preparatory schools in the modern sense. But the preparation in geometry was not thorough. The university did not continue the work from where the schools left it, but reviewed and enlarged upon it. We conclude therefore that in the first half of the nineteenth century the mathematics of the Latin schools was restricted to first principles.

Mathematics was also taught in the Gymnasia, the programs of which were regulated by the royal decrees of 1815, 1816, and 1826. The mathematical programs of this class of schools were more extended than those of the Latin schools, although both of these types of schools prepared for the university. Parvé[2] says that in the programs of the Gymnasia the limits of algebra and arithmetic were quite well defined, but in geometry there was nothing definite. As a consequence solid geometry disappeared from the programs. This lack of uniformity may explain why the university had to give its attention to the elementary work in geometry. Parvé also says that the methods in these schools were often defective. There was much learning by

[1] Similar plans of grading text-books are employed by the text-book writers of to-day.

[2] Cardinaal (*op. cit.*, p. 328) refers to a report of Parvé, inspector of secondary schools, in 1850.

heart, and the pupils were forced to demonstrate mechanically.

Parvé sought to unify the work in the preparatory schools and so he set definite limits to the elementary mathematics. For the subject-matter of geometry he recommended the following:[1] Fundamental notions, congruence and similarity of plane figures, circles, areas of plane figures, regular polygons, and the geometry of solids. Thus it was not until the middle of the last century that the subject-matter of geometry in the secondary schools of Holland assumed its present proportions.

Thus the teaching of geometry in Holland up to the beginning of the eighteenth century was essentially of a practical nature. During this time we find little reference to the teaching of Euclid, although the "Elements" went through several editions. During the eighteenth century there appeared some texts showing an independent treatment of logical geometry, and by the end of that century the teaching of geometry was well established in the secondary schools. Up to the middle of the nineteenth century the method of teaching was that of learning by heart, which, it is safe to suppose, continued even later.

OTHER EUROPEAN COUNTRIES

Geometry did not gain a foothold in secondary instruction in Austria[2] until the middle of the nineteenth century. During the sixteenth century the Jesuits dominated the field of superior education and mathematics was sacrificed to the benefit of the classics. After the expulsion of the Jesuits in 1773, petty reforms were introduced, but the teaching of mathematics was not yet systematized in the Gymnasia. Some of these, however, in 1775 offered courses in geometry that treated the subject in a practical yet scientific manner.[3] In 1805 there was a plan to teach mathematics in the two higher classes of the Gymnasia, but it was not carried fully into effect, for in 1841 geometry was not yet in the program of studies actually in use. But the idea was

[1] Cardinaal, *op. cit.*, p. 328.

[2] Simon, *L'enseignement des mathématiques au gymnase autrichien.* In Ens. Math., 1902, pp. 157-166; Baran, *Geschichte der alten lateinischen Staatschule und des Gymnasiums in Krems*, pp. 115-135; Beer und Hochegger, *op. cit.*, Vol. I, pp. 266-532; Paulsen, *op. cit.*, p. 695.

[3] Monumenta Germaniæ Pædagogica, 30, pp. 106-128.

being favored and the belief was getting stronger that the teaching of mathematics should be in the hands of specialists. In 1848 the Gymnasia were entirely reorganized. The course of study of eight years was divided into two courses of four years each. The first besides being preparatory was also a course in general culture. The last treated the subjects in a more scientific manner and prepared for the university. Before this time mathematics received one-half an hour a week. Now mathematics received three hours, and natural science, which had not been taught before, received the same number. The first teaching was characterized by emphasizing quantity rather than quality. By 1856 the teaching became more profound and systematic, and in 1884 the teaching of mathematics ranked equally with the languages and history.

Bulgaria[1] also had a late mathematical development. This began after the Turkish yoke was thrown off. It was not until 1839 that a work of any kind was printed in the Bulgarian language, the first geometry appearing in 1867.[2] It was not until 1850 that the Gymnasium programs gave geometry its place as a separate subject.[3]

In Switzerland, the study of geometry in the secondary schools was slighted until the beginning of the eighteenth century, when it began to take rank with the other subjects. In the Gymnasium at Basel, in 1717, under the influence of John Bernoulli, geometry, together with geography and history, occupied a place on the school program.[4] As Sturm's "Mathesis juvenilis" was used, one can judge that the work was of a character adapted to young minds.

THE UNITED STATES

As was the case in Europe, geometry was first taught in the United States[5] in the universities, and so continued until after the middle of the nineteenth century. Harvard College was

[1] Sourek, *L'enseignement mathématique en Bulgaria*, Ens. Math., 1905, pp. 257-270.

[2] By V. Gruev, printed in Vienna.

[3] Rein, *Encyklopädisches Handbuch der Pädagogik*, I, p. 804.

[4] Burckhardt, *Geschichte des Gymnasiums zu Basel*, pp. 87-319.

[5] Cajori, *The Teaching and History of Mathematics in the United States*. Unless otherwise stated, Cajori is the authority here referred to.

founded in 1636, at which time arithmetic and geometry were taught in the last year of the three years' course. One day a week was given to these studies together for three-fourths of the year. In 1655, when the four years' course was adopted, mathematics was still taught in the senior year. In 1726 printed texts began to appear, the first printed geometry used being that of John H. Alsted (1558–1638).[1] About this time Euclid was used at Harvard for the first time. By 1726 Yale University (founded in 1701), as well as Harvard, taught arithmetic and geometry in the senior year. Euclid was used as a text at Yale in 1733. In 1744 geometry was taught there in the second year instead of the fourth, and had the same position in 1777. This change did not occur at Harvard before 1787. It was not until 1818 that geometry began to be taught there in the first year. At the University of Pennsylvania (founded in 1755) the descent of geometry through the classes was more rapid, it being taught in the first year in 1758 together with arithmetic and algebra. The geometry included the first six books of Euclid in the first year, and in the second year Euclid XI and XII, together with plane and spherical trigonometry and applied mathematics.

It was not until 1844[2] that Harvard required geometry for entrance, and even then merely the elementary notions were demanded. In 1865 the demands were no more rigid, as can be seen from the title of the text in which the student was to be prepared.[3] Yale in 1855 followed Harvard's example by requiring two books from Playfair's Euclid, a higher standard than that set at Harvard. In 1887 all of plane geometry was required, nothing being said about Euclid.

The other colleges and universities that came into existence had varying courses and requirements for admission, but they approximated on the whole the standards set at Harvard and Yale. A particular exception was found in some of the institu-

[1] Ward's "Mathematics" was used between 1726 and 1738. The work was not strict in its logic.

[2] Broome, *A Historical and Critical Discussion of College Admission Requirements*, p. 45. Cajori gives the date 1843.

[3] The book was, Hill's Second Book in Geometry, Parts I and II, or "An Introduction to Geometry as the Science of Form." The student was supposed to study as far as p. 130.

tions in the South. The work in these universities was seriously hampered during the Civil War and their standards were necessarily lowered.

Although it was not until 1844 that the universities began to insist on geometry as a requirement for admission, it does not follow that previous to that time geometry was not taught in schools of lower grade. The academies which came into existence soon after the Revolutionary War were not at first preparatory schools, their courses of study even including some university subjects. At Phillips Exeter Academy, in 1818, geometry was taught in the fourth or highest class of the classical course. In the three years' English course geometry was taught in the second year, together with plane trigonometry and its applications to the mensuration of heights and distances. A separate course was also given during this year in the mensuration of surfaces and solids. The third year included the applications of mathematics.[1]

Geometry found a small place in the Colonial grammar schools. These schools before the middle of the eighteenth century were not preparatory schools for the universities, and the mathematics taught there "smacked of trade." Geometry was then taught with navigation and surveying.[2] One of the oldest grammar schools with a different aim was the Boston Latin School, whose function was to prepare students for college. During the period from about 1815 to 1828, geometry was taught in the fourth and fifth years of the five years' course.[3] The work seems to have been of a high standard.

The development of the American high school began at the end of the first quarter of the last century. The first of these was the English High School at Boston, founded in 1821. The movement soon spread and, according to the estimate of Dr. W. T. Harris, by 1860 there were forty high schools in the United States.[4] The first course of the Boston English High School comprised three years of work. In the second year, the program included geometry, trigonometry with applications to

[1] Brown, *The Making of Our Middle Schools*, pp. 230-238.

[2] *Ibid.*, pp. 128-134.

[3] *Catalogue* Boston Latin School, 1635-1885, with historical sketch, pp. 60-64.

[4] *Ibid.*, p. 313.

the mensuration of heights and distances, and the mensuration of surfaces, and solids.[1] The character of this course indicates a preparation for immediate practical service. The courses at the university during this same period had much in common with the above.

By 1844, when the universities began to place geometry on the list of entrance requirements, the high schools naturally took up more seriously the teaching of the subject. Further stimulus was given to the work when the University of Michigan in 1871 inaugurated the system of accrediting schools. This system permits students to enter the university without examination after the university has inspected the work in the high schools. The system was adopted by other state universities in the west and is now an established institution. The east has been slow in adopting this plan, the older institutions in particular still admitting by examination only.

The mathematical teaching in the universities was at first influenced by the English, although it is not expressly stated that Euclid was employed as a text until Harvard was nearly 100 years old. The influence of the English increased until the invasion of French mathematics, which began in the first quarter of the nineteenth century.[2] In the programs of the universities, Legendre now began to take the place of Euclid. As we have seen, most of the early texts used in the universities were either the English Euclids or works based on them, the editions of Robert Simson and Playfair being in common use. After the introduction of Legendre, the use of Euclid necessarily decreased until to-day the books of the English type are rarely employed. Some of the most popular geometries that have been based on Legendre are those of Davies (1840) and Chauvenet (1870).

Logical rigor in method was not required before the middle of the eighteenth century, if we are to judge from the nature of a text in common use at that time. Ward's mathematical books were in use at both Harvard and Yale. His geometry begins with definitions, followed by twenty problems on constructions. Then follow some theorems demonstrated in a loose fashion.

[1] Catalog Boston Latin School, pp. 296-301.

[2] The influence of the French began when Claude Crozet was appointed professor of mathematics at the West Point Military Academy (1816-1823).

The treatment of parallels in particular is not sound. The way geometry was taught at Dartmouth soon after its foundation in 1769 is seen from a statement of Samuel Gilman Brown: "I remember hearing one of the older graduates say that the first lesson of his class in mathematics was twenty pages in Euclid, the instructor remarking that he should require only the captions of the propositions, but if any doubted the truth of them he might read demonstrations, though for *his* part his mind was perfectly satisfied."[1] As Ward's book was used at Yale as late as 1777, we may safely judge that Euclid's logic had no very strong hold on the teaching. But during the last quarter of the eighteenth century, when English influence became stronger, Euclid was taught more generally, and hence the teaching of geometry must have been characterized by greater logical rigor.

To summarize briefly, the development of the teaching of geometry in the United States up to the last quarter of the nineteenth century has been as follows: The universities first took up the work and did not generally give it over to the secondary schools until the middle of the last century. Some of the smaller institutions did not make geometry an entrance requirement until about twenty years ago. During the Colonial period, geometry was studied in the grammar schools, and after the Revolution in the newly founded academies. After 1821, when the first high school was established in Boston, geometry found a place in these schools. The teaching in the early universities and in the other schools was at first quite practical. Logic became of more importance when the English Euclids were in greatest favor, during the last quarter of the eighteenth and the first quarter of the nineteenth centuries. Notwithstanding the fact that the French influence, which began about 1817, has tended to make the teaching again more practical, the English influence has been lasting. Excepting for a form of dogmatism that characterized some of the early teaching, the demonstrative method has been the common practice.

The following general conclusions may be drawn in regard to the teaching of geometry since the beginning of the sixteenth century: During the sixteenth and seventeenth cen-

[1] Cajori, *op. cit.*, p. 74.

turies but slight attention was given to the study of geometry in the secondary schools of Germany. In France it received even less attention. In England we find no mention of its being taught outside the universities. In Russia and the United States, secondary schools were not yet created. By the end of the seventeenth century, the teaching of geometry was beginning to be somewhat systematized in Germany, but there was little intensity in the study, general knowledge being the chief desideratum. The geometry taught was largely in connection with geography and surveying. Euclid was thought too hard and was not looked upon with favor.

The eighteenth century showed a change in the teaching of geometry and of mathematics in general. Academic texts now began to be used in the schools. Hitherto geometries were mostly either practical, with little reference to logic, or editions of Euclid, the opposite extreme. This combination of the logical with the practical was especially marked in France and Germany. In England, the books based on Euclid were getting to be more academic in character. In this century geometry was more commonly taught in the secondary schools, this being specially true in Germany, although some of the schools still taught it only as an elective. Simple construction work and exercises in practical problems found their way into the Realschulen in Germany. In France the study of geometry was emphasized in the military schools, but the eighteenth century showed a great development also in the colleges, and by the end of that century geometry and mathematics in general were flourishing in those institutions. In 1794 Legendre, following the example of his less illustrious predecessors in France and in Germany, composed his "Éléments" on lines different from those of Euclid. There is some probability that Euclid was being taught in the secondary schools of England by the middle of this century. In Holland and Switzerland the Gymnasia saw the work being systematized. In Russia, the Gymnasia were being created and were beginning to be interested in the study of geometry as a science. In the United States, the eighteenth century still saw geometry taught only in the universities.

In the nineteenth century the teaching of mathematics in the secondary schools was more completely systematized, this being

shown in the working out of the curriculum and in the limits assigned to the selection of subject-matter. Parallel with this the study of mathematics has been carried on with greater intensity, which has resulted in a certain isolation in the several mathematical branches. In the United States and England this is especially noticeable, while in Germany and France the relation between the several branches has received some recognition. We have shown that for the most part the teaching of geometry in the early secondary schools of the various countries was associated with its applications in surveying. The tendency in the nineteenth century has been away from any such applications. It is only within the last few years that we hear of demands being made for a closer relationship between the teaching of pure and applied mathematics in the secondary schools.

The secondary schools of the United States began to prepare more generally for the university by the middle of the nineteenth century. In England, also, the "Public Schools" were beginning to require the teaching of Euclid, which means that those schools were now preparing in geometry for the university. In Germany, before this time, the Gymnasia were given the sole privilege of thus preparing students. We can therefore see a special reason why the study of mathematics has become more intensive.

As regards changes in method, it is hard to say just when demonstrative work, as we have it to-day, began. The early Greeks employed the Socratic method in their teaching. Euclid showed the world a system of demonstrative geometry. In the early universities, the pupils learned from dictation or lectures. In the secondary schools in Germany in the sixteenth and seventeenth centuries similar methods were employed. The students usually learned the work by heart and recited it. During the seventeenth century in Germany, demonstrative work, as opposed to working by rule, began to be more emphasized, and in the next century the custom of explaining propositions was common. In Russia the eighteenth century showed this same transition. The work previously had been taught dogmatically, the students learning by heart and working by rule. Now they were taught to demonstrate their propositions and apply them rationally. In England the students practiced the demonstrative method in their study of Euclid. Though the nineteenth

century saw the dogmatic method formally discredited, yet traces of it remained in those countries where the text-book has been of prime importance and hence where the pupils have had the tendency to learn by heart. England and the United States especially come under this heading.

In this development three educational aims have been apparent, the practical, the logical, and that of preparing for advanced work in mathematics. These have been associated in greater or less degree in the various countries and institutions considered.

CHAPTER VI

PRESENT-DAY TEACHING OF GEOMETRY

This chapter will deal primarily with the teaching of geometry in secondary schools. In the United States the term secondary school is commonly applied to the high school. It represents the last four years' work in a twelve years' course. All students take the same work during the first eight years whether they intend going into the high school or not. In England, France, and Germany the secondary school is not the continuation of the elementary school as with us, but a separate institution which prepares usually for general culture or the "higher callings." The boy generally enters these institutions in those countries at the age of nine and remains nine years.

GERMANY

The three classes of secondary schools in Germany[1] are the Gymnasia, the Realgymnasia, and the Oberrealschulen. These all have courses of nine years. In some of the smaller cities the last three years are omitted and the institutions are called the Progymnasia, the Realprogymnasia, and the Realschulen. The Gymnasium is the classical institution with both Latin and Greek. The Realgymnasium does not offer Greek, and the Oberrealschule offers neither Latin nor Greek. The last two institutions lay more emphasis on mathematics. In the Prussian Lehrplan[2] for 1901, it is prescribed that the Gymnasia give thirty-four periods per week to "Rechnen" and mathematics, the Realgymnasia forty-two periods, and the Oberrealschulen forty-seven periods.

[1] For an interesting account of the present teaching of mathematics in Prussia, see Young, *The Teaching of Mathematics in the Higher Schools of Prussia.*

[2] *Centralblatt für die gesammte Unterrichtsverwaltung in Preussen,* 1901 p. 473ff

In the Gymnasia, geometry is begun in the third year (*Quarta*), when the pupil is eleven or twelve, the work being of a propædeutic character. Two periods a week are given to this work. Here the student is made familiar with the fundamental conceptions of straight line, angles, and triangles, and does construction work with the compasses and rule. By the end of the seventh year (*Obersecunda*), the student has completed plane geometry, and during the next two years has completed solid geometry. Algebra is begun in the fourth year and continued throughout the course. Trigonometry is begun in the sixth year and also continues parallel with the other mathematical subjects. The mathematical course is practically a unit, geometry being begun first. Then algebra enters and finds application in the geometry. Trigonometry is begun when similar triangles are being studied in geometry, and the two subjects are made mutually dependent on each other.

In the Realgymnasium, geometry is begun also in the third year, but more time is given in this school to mathematics, so we find solid geometry entering into the work in the sixth year, when the boy is about fourteen years of age. In the Oberrealschule the first work in geometry is given in the second year (*Quinta*). The work on the whole is like that in the Realgymnasium, but in the higher classes it is more extensive. Plane geometry is studied even after solid geometry is begun in both of these latter schools, but the work is not the standard Euclidean geometry, for it includes the theory of harmonic points and rays, and symmetry.

The schools in Frankfort-on-the-Main do not follow strictly the above programs. In the Goethe Gymnasium, for instance, we find these differences: Geometric drawing and mensuration are begun in *Quinta*. Solid geometry is begun in *Untersecunda* and taught for four years. Plane geometry is also continued and becomes a study of the conic sections, analytically considered. The work more nearly coincides with that of the Realgymnasium described above.

The first work in geometry in the German secondary schools is entirely propædeutic, beginning in *Quarta*[1] or *Quinta* according to the school. The pupil learns here how to use the proper in-

[1] In the Goethe and Kaiser Friedrich Gymnasia in Frankfort, the writer observed very rigid logical work in *Quarta*.

struments and gains conceptions of the different geometric figures. The work of the following year links very closely with this preparatory work, but the logical now assumes prominence. The course includes the subject-matter of Euclidean geometry, but Euclid as a text has no place in the German schools. In fact, very little dependence is placed upon any text. The value of modern geometry is being recognized, for in the programs mention is made of harmonic points and rays, and the theory of transversals, and algebraic and trigonometric methods are freely applied to the work in geometry. In fact, there is such an interlacing of the work that one sometimes finds it difficult to name the subject that is being taught.[1]

In the German schools, the classroom is a place for instruction, not a place where the pupils "say" their lessons. The lesson in geometry begins by the teacher calling a boy to the single small blackboard behind the teacher's desk. The figure is drawn, the teacher meanwhile questioning either the boy at the board or the other members of the class. A single student rarely gives an entire proof except in resumé. The teacher is careful that all members of the class participate, the method being that of class instruction as opposed to individual instruction. The work is very thorough, if necessary the whole hour being given to a theorem with its applications and corollaries. Time is not lost, for the students understand the work, and when the foundations are once laid they are secure. Very little homework is assigned, the exception generally being in connection with some construction problem. With teachers who know their subjects, who know how to economize time, and who know how to instruct, the mathematical education of the German boy is well provided for.

FRANCE

The lycées[2] are the great institutions for secondary education in France. They correspond to the German institutions just described, but with this difference: In Germany, the courses vary with the different institutions, while in France, as in the United States, one institution gives a choice of several courses.

[1] This is particularly true of the upper classes.

[2] In the smaller cities are found "collèges," which are also secondary institutions, but the lycées are of higher standard.

The course of study for the lycées and colleges given in the decree of May 31, 1902,[1] instituted the new plan of dividing the course of study into two cycles of four and three years each. In the first cycle the pupil has the choice of two courses. In the A course, Latin and Greek are obligatory. In the B course, science and drawing receive greater attention, Latin and Greek not being required. In the second cycle, extending through three years, there is a choice of four courses, namely: A, Latin and Greek; B, Latin and a Modern Language; C, Latin and Science; and D, Modern Languages and Science.

We shall consider first the two courses of the first cycle. In the second year[2] (*cinquième*) of the scientific course (B),[3] the pupils perform geometric constructions by means of the set-square, the rule, the compasses, and the protractor. They study the properties of triangles, parallelograms, circles, and other plane figures, but strict logical work is not yet begun. In the *classe de quatrième*, the logical work receives more emphasis. By the end of this, the third year, the main part of the subject-matter of plane geometry is completed. In connection with the treatment of similar triangles, the definitions of the trigonometric functions are introduced. The construction of simple curves like the cissoid and conchoid is undertaken toward the end of the year. In the fourth year (*troisième*), solid geometry is studied together with some work in surveying.

The classical course (A) of the first cycle gives a much more limited course in geometry. The work is begun a year later[4] than in the scientific course. In the fourth year plane geometry is finished except for some of the work in mensuration.

In the second cycle of three years, the mathematics is the same in the A and B courses as it is in courses C and D, where it is emphasized the most. In the first year (*seconde*) of this cycle, in the A and B courses (the classical and literary), solid

[1] *Plan d'études et programmes d'enseignement dans les lycées et collèges de garçons* (Arrêtés du 31 Mai, 1902).

[2] As in the Gymnasia of Germany, the classes are numbered opposite to the method in the United States. In the *classe de cinquième* the pupils are about twelve years of age.

[3] The plan of 1902 as modified by the new program for mathematics of September, 1905. See Ens. Math., 1906, pp. 65-77

[4] In the third year (*quatrième*).

geometry is studied. In the second year (*première*), further work is given in plane geometry involving metrical relations. Here the fundamental notions of trigonometry are introduced and solid geometry is completed. In the third and last year (*classe de philosophie*), no geometry as such is studied, attention being given to advanced algebra and its applications in analytic geometry.

In the C and D courses of the second cycle, the mathematics is more extensive than in the courses just described. In the first year attention is again given to plane geometry. The previous work is reviewed and expanded. The notion of geometric locus is introduced for the first time and trigonometry is further developed. The practical work of the year ends with applications in simple surveying. In the second year of the second cycle, solid geometry is again studied and completed. Trigonometry for the first time becomes a separate subject. The last year (*classe de mathématique*) provides for a review of the previous work and introduces phases of modern geometry. Higher studies in mathematics are also begun in this year.

We notice that in the French as in the German schools geometry extends through several years. It is begun in the *classe de cinquième* in the scientific course, when the boy is eleven or twelve years old, and extends six years, to the completion of the second cycle. Solid geometry and algebra are begun in the *classe de troisième* and both continue for four years. Trigonometry is begun at first in connection with geometry in the third year of the first cycle in the scientific course and continues to the end, a period of five years. We thus see the same interlacing of the mathematical subjects as in the German schools. As in the German Realgymnasium and Oberrealschule, the mathematical course in the French lycée easily includes the work of the first year of an American college.

Euclid has no place in the French schools. Even the use of Legendre as a text is seen only here and there, the teachers preferring to use texts of their own or of a colleague. The first work in geometry is practical and the transition into the logical is quite gradual, but when the logical standards are once established, the teachers are very exacting with their pupils. When the more scientific study of geometry is begun, the algebraic method is employed whenever opportunity offers. In like man-

ner trigonometric methods are used after proportion is reached. The books used do not always show this interlacing of the subjects.[1] The teachers emphasize this particularly in the exercise work.

As in the German schools, the class and not the pupil is the unit for instruction. At the first of the hour the teacher returns the note-books, which contain the previously assigned written work (devoirs). If no great difficulties are encountered only a few comments are made on the corrected work. If the assigned work be difficult and the students have not done well, the work is all carefully explained in class. As in the new work, a pupil passes to the one small blackboard, and under the questioning of the teacher, the exercises are explained by the class. This correcting of the written work may take the whole hour. The teacher is very thorough and does not sacrifice quality for quantity. As a rule it takes but a few minutes to correct the assigned work, and then some new exercises are assigned for the next week. Several pupils may pass to the board during the hour. The pupils invariably explain what they are doing while they draw the figures. The figure being drawn, the teacher seeks to bring the whole class into the work. One feature of schoolroom practice, however, tends to suppress the spontaneity of the class. Whether the teacher be explaining, or the pupil reciting, it is the practice of the class to keep taking notes. The result is that quick, sharp work on the part of the pupils is frequently lacking, so that the teacher has to do a great deal of the talking. One good feature of the classroom method is the care of the teacher to bring out always the practical side of the work, but this is limited to the field of mathematics. There are no applications of geometry to field work in surveying or to the other sciences.[2]

A movement in France for the teaching of solid geometry in connection with plane geometry has gained considerable headway

[1] This statement, like much of the description of the French schools, is based on personal observation in the lycées of Paris. It is assumed in the above that the work there is typical.

[2] Based on data obtained from the lycées in Paris in June, 1905. In the last year this application to science is made after the subject-matter of elementary geometry has been completed. The new course, in effect September, 1905, provides for the applications of geometry to surveying before the course in geometry is completed

in recent years. The "Nouveaux éléments de géométrie" by Professor Charles Méray, which appeared in 1873, has, after an interim of over twenty years, claimed the attention of French teachers of mathematics. Méray, however, is not the first to have originated this idea.[1] Gergonne, in about 1825, raised the question if it were natural to separate the teaching of solid geometry from that of plane.[2] Before Gergonne, Lacroix in 1816 pointed out the analogies between certain theorems of plane and solid geometry and suggested that it is possible to combine this teaching in certain cases.[3] Mahistre in 1840 pointed out the advantages of employing this principle of analogy in the teaching of geometry.[4] Valat in 1866 published a pamphlet on the reforms of teaching elementary geometry.[5] In this he gives a syllabus which shows this principle of analogy, and by the sequence of the books one can see the idea of "fusion" fully carried out. Books I, III, V, and VII are on plane geometry, and Books II, IV, VI, and VIII on solid geometry. Book II in solid geometry embodies subject-matter analogous to that in Book I of plane geometry, and so on with the rest. For example, Book III contains in Chapter I, circles, arcs, chords, and measure of arcs. Chapter I of Book IV treats the sphere and its general properties. Chapter II of Book II is on the intersection and contact of circles. Chapter II of Book IV, on the intersection and contact of two spheres. Chapter III of Book III, similar figures. Chapter III of Book IV, similar solids. Chapters IV in both books treat of areas of figures. The author ranges his subject-matter on the page in two parallel columns, the books of plane geometry on the left, those of solid geometry on the right, with the books and chapters in both systems set over against each other. In such a scheme, by studying the odd numbered books, plane geometry could be studied first, or after each book in plane geometry

[1] It may be interesting to recall that Menelaus proved a theorem in plane geometry as a lemma to the one corresponding in spherical geometry. See above, pp. 36-37.

[2] Loria, *Sur l'enseignement des mathématiques élémentaires en Italie*, Ens. Math., 1905, pp. 11-20.

[3] See above, pp. 84-85.

[4] See Loria, Ens. Math., p. 15; L. Ripert in Ens. Math., 1899, p. 62.

[5] Valat, *Plan d'une géométrie nouvelle ou réforme de l'enseignement de la géométrie élémentaire.*

the corresponding book in solid geometry could be studied, or after each chapter in a particular book in plane geometry the corresponding chapter of the corresponding book in solid geometry could be taken up.

Méray does not set chapter over against chapter or book against book, as Valat did in his syllabus. He has, instead, set individual propositions in solid geometry over against the corresponding ones in plane geometry. Thus he treats in close connection the properties of the straight line and of the plane; parallelism of straight lines and planes and the theorems on their intersection; perpendicularity of straight lines and of planes; comparison of plane and dihedral angles. But since Méray did not of course always find it possible to formulate a complete parallelism between the two geometries, some of his chapters treat the geometries separately.

By the fact that a considerable number of schools have adopted Méray's book, it appears that the "fusion" movement is making some progress in France. In a recent article,[1] Professor Méray says that the "Nouveaux éléments" is now employed exclusively or partly as a text in thirty normal schools, and is also in use in some of the superior primary schools. The opinion of a professor who has used the above text is worth citing. M. Billiet introduced the book in the normal school at Auxerre in 1898, and again used it in 1899. Previous to the use of the new method, he says,[2] "The beginning was slow, the demonstrations tiresome for the pupils, and often entangled with useless details; the spirit of the pupils was hampered by useless, at times pedantic, quibbling; under the pretense of making a complete proof, they were foolishly delayed, in order that they might follow a uselessly exact method." Regarding the use of the new book M. Billiet says: "The lessons are animated and the pupils are interested in the new method. . . . The pupils of the first year follow the new course. They advance surprisingly, their intelligence is lively, and they work very easily because of the new light which is presented to them. All are impressed with the sequence of theorems and the simplicity of the demonstrations." In a

[1] Méray, *Justification des procédés et de l'ordonnance des nouveaux éléments de géométrie*, Ens. Math., 1904, pp. 89-123.

[2] Perrin, *La méthode de M. Méray pour l'enseignement de la géométrie*, Ens. Math., 1903, pp. 441-446.

second report (1901) of M. Billiet, the following conclusions are given: "(1) The simultaneous teaching of plane and solid geometry saves time. (2) The new method restores the agreement between the various subjects in the mathematical program and those of theoretical and practical teaching with which it is connected. (3) It appeals to intelligence more than to memory. (4) It accustoms the pupils to think for themselves. . . ." From the above we see at least what is being claimed for the new method.

It appears that Méray's method has received official recognition in France. In *L'Enseignement Mathématique*,[1] the editor quotes from a circular written by Professor C. Bourlet of the Lycée St. Louis, addressed to his colleagues. M. Bourlet says that in August, 1904, the French Association for the Advancement of Science voted in favor of the adoption of Méray's method. This was transmitted to the Minister of Education. As a result, "the instructions annexed to the new decree of July, 1905, invites, among other things, the teachers to follow a method entirely new in geometry. . . ." "It is that with joy," writes M. Bourlet, "that I have undertaken the delicate task of rewriting the 'Nouveaux éléments de géométrie,' so as to conform to the program, to the instructors, to the ideas of M. Méray, and to the necessities of our teaching."

For two reasons this new movement in France seems to be of a permanent nature. First, it has already been tried, apparently with success, in many of the schools. Secondly, it has received ministerial sanction. It is to be noted, however, that the lycées have not made use of Méray's text. Undoubtedly the problem of preparing for the higher examinations has influenced the teachers in these schools. It remains to be seen if the new book to be prepared by M. Bourlet will find a place in the lycées.

ITALY

Before 1848, the classics reigned supreme in Italy. With the political changes that came in that year, the plan of secondary instruction was entirely reorganized. By the law of Nov. 13, 1859, which gave direction to this new organization, the classical schools were divided into a lower and a higher grade. The

[1] Nov., 1905, pp. 488-489.

school of lower grade is the *ginnasio* with a five years' course, and that of the higher grade the *liceo* with a three years' course. By this same law of 1859, two grades of technical schools were created, the *scuola tecnica* and the *istituto tecnico*.[1] The first is the more elementary, the latter offering the real scientific training. The classical schools above mentioned prepare for the university studies and the "higher callings," while the aim of the technical schools is "to discover and strengthen those properties which develop the commercial and industrial life of a nation."[2]

Of the classical schools, pupils are admitted into the ginnasio at the age of ten or eleven. During the first three years only two hours a week are devoted to mathematics, and throughout the five years of the course the work is not extensive.[3] But in the three years of the liceo the work is much broader. It comprised in 1880 the geometry of Euclid, arithmetic, algebra, logarithms, trigonometry, and stereometry.[4] The course in geometry in the licei in 1884 was: For the first year, the first four books of Euclid; for the second year, Books V and VI; and for the third year, solid geometry taught from a modern treatise.[5]

Up to 1904 there was one course for all students in the classical schools. By the decree of Nov. 11, 1904, the Minister of Public Instruction introduced an essential change.[6] During the last two years of the liceo the student can now choose between two courses, which emphasize Greek and mathematics respectively.[7] The course in geometry, beginning with the

[1] Schmid, *Encyklopädie des gesammten Erziehungs*, article *Italien*, p. 744. For a general description of these types of schools, see Hippeau, *L'instruction publique en Italie.* For an account of the educational development of Italy in the last century see the article on Italy in Baumeister, *op. cit.*, I_2, p. 541

[2] Schmid, *op. cit.*, p. 744

[3] Degani, *Some Aspects of Italian Education*, pp. 26-27

[4] Schmid, *op. cit.*, p. 746

[5] Candido, *Sur la fusion de la planimétrie et de la stéréométrie dans l'enseignement de la géométrie élémentaire en Italie*, Ens. Math., 1899, pp. 204-215. Hereafter referred to as Candido.

[6] Battazzi, *Un essai de réforme des études moyennes classiques en Italie*, Ens. Math., 1905, pp. 400-406.

[7] This elective principle has met with strong opposition from teachers of mathematics in Italy. At the meeting of the association "Mathesis" held in Milan, April 21 and 22, 1905, it was resolved that the association feels that a single program of elementary mathematics should be obligatory for all pupils and for all classes of the liceo. Battazzi, *op. cit.*, p. 403.

last two years of the ginnasio, comprises plane geometry, including general conceptions, equality of figures, and some notions of equivalence. In the first year of the liceo, the work includes relations of positions, equality of solids, proportion and similarity of plane figures, theory and application of mensuration to plane figures, and practical rules for measuring solids and surfaces not plane. In the second and third years are taught the equivalence and similarity of solids, theory of the measure of surfaces not plane and of solids, and the application of algebra to geometry. Algebra and trigonometry are begun the first year of the liceo, after the introduction of geometry.

The geometric traditions in Italy have on the whole been Euclidean. In ancient Piedmont, where the French influence was strong, Legendre was preferred to the rigor of Euclid, and the provinces which threw off the yoke of Austria used books written with a commercial aim. These were not satisfactory, in particular to Cremona, who was then teacher in a secondary school in Lombardy. Not wishing to adopt such books, he urged the use of better ones. He was appointed in 1867 by his government "to lay out the general lines of a reform in the teaching of geometry in the classical schools." He immediately suggested a return to Euclid, pure and simple, and in this he was supported by Brioschi and Betti, who had edited an edition of the "Elements."[1] The government acted on the recommendation of Cremona.[2] The program of 1884[3] shows the results of this recommendation, but, judging from the character of important texts in use about that time,[4] we are not to believe that Euclid was strictly adhered to. As for the teaching of geometry at the present time, the

[1] Loria, *op. cit.*, Ens. Math., 1905, pp. 11-20.

[2] In the preface of Sannia and D'Ovidio's geometry (1888, 7th edition) is found the statement that in 1867 the Italian government had ordered a return to Euclid. The practical tendencies in Legendre are also discouragingly mentioned.

[3] See above, p. 113.

[4] The sequence in the geometry of Sannia and D'Ovidio is practically that of Legendre. Numerical work is found in mensuration and algebra is employed in proportion. The book of Faifofer (1887), as the preface states, is intended primarily for use in the technical and normal schools. The work is introduced from the practical standpoint and proportion is at first treated arithmetically. Loria (Ens. Math., p.12) states indirectly that this text was used in the licei.

program of 1904, already referred to, shows no close adherence to Euclid. The sequence of subject-matter, in particular, shows a marked departure from the "Elements." That the actual teaching is away from Euclid is seen from a statement of Professor Loria. He says,[1] "The last of the Italian mathematicians who have written on elementary geometry are Enriques and Amaldi, who have attached the new to the traditional Euclid, and propose in their recent book to have a text that fits the actual conditions in the schools."

Italy, like France, has in recent years been agitating the simultaneous teaching of plane and solid geometry.[2] The text of De Paolis, which appeared in 1884, embodies this aim. Although the book was somewhat beyond the capacity of immature students, some young teachers, "braving critics as well as difficulties,"[3] had the temerity to adopt it in some of the licei. A little later appeared the geometry of Professor Andriani, which also used the method of fusion. According to Candido,[3] Andriani went too far and forced analogies between plane and solid geometry. A pupil of De Paolis, Dr. Guilio Lazzeri, professor in the Royal Academy of Leghorn, adopted the new method and influenced his colleagues to do likewise. He wrote out a course himself in 1887 and another in 1889. This was tested in teaching and resulted in the "Elementi di geometria" by Lazzeri and Bassani, which appeared in 1891. Candido says[4] that, since the work was more practical and serviceable than the one by De Paolis, it received favorable criticism and was adopted in many licei. The text comprises five books. The first three are independent of number and treat the properties of position and of magnitude in the plane and in space which are derived from the fundamental notion of equality. Book IV is on the theory of equivalent magnitudes. It is also free from number. Book V treats the theory of proportional magnitudes and of measures. Number is here involved.

Since the above book first appeared, the "fusion" idea has found place in the texts of Professor Veronese (1897) and of Pro-

[1] Loria, Ens. Math., p. 13.

[2] *Ibid.*, pp. 11-19; Candido, pp. 204-215.

[3] Candido, p. 206.

[4] *Ibid.*, p. 207.

fessor Reggio (1898).[1] Attention has also been given to this question by the *Association Mathesis*,[2] which organization has for its aim the betterment of the teaching of mathematics. At its meeting in 1896 one of the first questions discussed was that of "fusion."[3] This subject was discussed in later meetings, the idea being to give the teachers a choice of the two methods in their work. In 1898 the association voted in favor of the new movement.[4] On the whole this movement seems to have progressed in Italy almost to the extent that it has in France. In fact, judging from the number of texts in Italy embodying the idea of "fusion," in this respect the latter country has gone beyond France. In both countries, prominent teachers' associations have passed favorable votes on the question, and in France the Minister of Education has decreed in favor of its use in the lycees.[5]

RUSSIA

The last thirty years of the nineteenth century saw the reorganization and extension of the Russian public school system.[6] The secondary system for the education of women

[1] Candido, p. 207.

[2] An organization of secondary school teachers of mathematics, existing since 1895.

[3] Professor Henri de Amicis read a paper on "Opportunities in Teaching for the Fusion of Plane Geometry with Solid Geometry." See Candido, p. 207.

[4] Candido, p. 211.

According to Ripert (Ens. Math., 1899, p. 62) the association gave recommendations to the Minister of Education in May, 1897.

[5] In England, Germany, and Italy it is being advocated that the course in elementary mathematics be enriched by bringing in phases of practical higher mathematics. Professor Perry stands for this very thing in England (see below, pp. 126-128), Professor Loria in Italy (see Battazzi, *op. cit.*, pp. 405-406), and Professor Klein in Germany (see Klein, *Über den mathematischen Unterricht an den höhern Schulen*, in Zeitschrift für mathematischen und Naturwissenschaftlichen Unterrichts, 1902, pp. 114-125).

[6] Bobynin, *L'enseignement mathématique en Russie. État actuel. Enseignement primaire.* Ens. Math., 1899, pp. 420-446.

Bobynin, *L'enseignement mathématique en Russie. État actuel. Enseignement secondaire.* Ens. Math., 1903, pp. 237-261.

For the more general features of education in Russia, see Hippeau, *L'instruction publique en Russie;* Beer und Hochegger, *op. cit.*; and the article on Russia in Baumeister, *op. cit.*, I_2, pp. 561-576.

was organized in 1870. About this time (1872) the systematic organization of schools for the training of teachers for the elementary schools was brought about. Two kinds of normal schools, which already existed, have furnished the necessary discipline. These are the Seminaries for Teachers and the Institutes for Teachers, the first of which prepare teachers for the city schools, the second for the schools of the villages. The latter schools are of lower standard than those of the cities. Further advance in public education was made in 1872 by the organization of the Realschulen and Progymnasia. In 1874 the primary schools were reorganized, and in 1889 inferior and secondary technical and trade schools were established. Geometry finds a place in all these institutions.

In the primary schools, practical geometry is studied only in the upper classes of the schools of the second class.[1] Here are taught the fundamental properties of plane figures, surveying, and linear drawing. Some of these schools extend their work to the practical study of the familiar solids. The inductive method predominates.

In the intermediate schools, the course extends through six years. Observational geometry is begun in the third year by the study of solids, little attention being given to plane figures. In the fourth year the study of the previous year is treated more in detail and the whole of plane geometry is covered in a general fashion. The practical work is emphasized, finding an application in surveying. In the fifth and sixth years, plane geometry is more fully developed. As for the logical development in these schools there are no demonstrations in the third year, but such work follows directly in the next. As compared with the work in the secondary schools, there is less theory and more practice.

The Seminary for Teachers in its three years' course covers plane and solid geometry with applications to surveying. The Institute for Teachers covers the same subject-matter, but accomplishes this in the first two years of its three years' course. In the former school the work of the first year includes a study of the geometry of the simpler plane figures, and surveying. In the second year, proportional lines, similar figures, regular poly-

[1] Those of the first class have a course of three years. Those of the second class, four or five years.

gons, measure of areas, and surveying. In the third year, straight lines and planes in space, polyhedra, measure of surfaces and of volumes, and surveying. During this year methods of teaching geometry in the elementary schools are emphasized.

Concerning the methods employed in these schools, Bobynin refers to two "notes" on methodology: "1. The systematic course is preceded by an examination of geometric solids. This is for those who have had no geometric training, but it is also a benefit for the majority in giving them models for future lessons in the school. 2. A new theorem is attacked by the method of analysis, participated in by the whole class. But when a theorem already studied in the class is demonstrated, synthesis only is used, as the analytic method is more difficult for pupils in their early work." Analysis, however, is very little used in the school work.

In the public secondary schools for boys, geometry is begun in the fourth class.[1] It is taught five times a week in the Realschule, and twice a week in the Gymnasium and Progymnasium.

The Realschule accomplishes more work than the other schools, the additional subject-matter including proportion and the similarity of triangles and polygons. In the fifth class[2] of the several schools, plane geometry is completed and solid geometry is begun. Solid geometry is completed in the sixth year, and in the Realschule all of geometry is reviewed. In the seventh year no geometry is taught, but in the eighth year of the Gymnasium a general review of the previous work in mathematics is prescribed. Algebra is begun in the third year, a year before geometry, and continues through the seventh year. Trigonometry is begun in the sixth year and extends for two years. This sequence differs from that in the German schools. In the latter, geometry is begun a year before algebra, and trigonometry is introduced early enough to be of service in plane geometry.

We thus see that with the reorganizing of the Russian school system on modern lines, geometry receives its full share of attention. In the primary and intermediate schools, practical geometry alone is taught. In the normal schools, where teachers

[1] In Russia, the system of numbering the years is like that in America. The highest class is the eighth.

[2] Four lessons per week in the Realgymnasium and two per week in both the Gymnasium and Progymnasium.

are trained for the primary schools, demonstrative geometry is taught, but the practical aspects of the subject are ever kept to the front. In the secondary schools for boys, the work covered corresponds to that in the best American high schools.

It has taken the Russians a long time to drive the dogmatic method of teaching from the schools. It is only in recent years that the demonstrative method has predominated. Even now the method of analysis is not sufficiently recognized.

HOLLAND

The secondary schools of Holland[1] were organized in 1663. They are of two kinds,[2] one class of schools offering a course of five years and the other one of three. The secondary school of five years is the principal institution for the teaching of elementary mathematics. It has a double aim, being a finishing school and at the same time preparatory. It prepares students particularly for the polytechnic, and as a consequence the work in geometry is very thorough. Trigonometry is introduced before solid geometry is begun. The secondary school of three years is not a preparatory institution, and hence the mathematics is more concise and less profound in details. The above schools do not include the Gymnasia. The latter appear originally to have been final schools, but since the creation of the so-called secondary schools, the Gymnasia have lacked more and more the character of final schools in order to assume that of preparatory schools for the university. We are to understand then that the "secondary schools" of five years prepare for the polytechnic and the Gymnasia for the university. Naturally the geometry in the Gymnasia would not be so practical as in the other class of schools. Two courses are offered in the Gymnasia, the literary and the scientific. Cardinaal says,

[1] Cardinaal, *L'enseignement mathématique en Hollande*, Ens. Math., 1900, pp. 307-309.

For the general features of education in Holland, see the article on Holland in Baumeister, *op. cit.*, I_2, pp. 671-672.

[2] There are two classes of primary schools, those fitting for the secondary schools and those functioning as final schools. By a law of 1878, practical geometry was prescribed for the latter schools, but in 1889 the work was discontinued and free-hand drawing was substituted.

"In examining the second course and comparing it with that of the secondary school, we see that descriptive geometry is replaced by spherical trigonometry and the first elements of analytic geometry, changes explainable when one thinks of the difference between the studies at the university and those at the polytechnic."[1] We see that the course in advanced mathematics in the Gymnasia is not unlike that in the German Gymnasia, while the so-called secondary schools in Holland seem to be more like the German Oberrealschulen, only the work is not so extensive.

The teaching of geometry on the whole appears to be associated with the preparation for advanced work in mathematics. In the secondary school of three years alone is it taught without reference to such a preparation.

ENGLAND

We have seen that elementary geometry began to be taught in the secondary schools of England in the early part of the nineteenth century, but not in any general way much before the middle of that century. One type of these schools is the great Public Schools, like Eton and Rugby. Another type is the Grammar Schools. The latter must not be confounded with the higher grades of the American elementary schools, for they differ little from the Public Schools in the character of their work. The Public Schools draw their patronage from the wealthy and aristocratic classes, while the Grammar Schools draw theirs from the middle classes.[2] The English secondary school bears no relation to the elementary school in the sense that obtains in the United States. In the elementary school, children enter at five years and finish at the age of fourteen. The curriculum is shaped with the idea that school life ends then. The secondary school receives boys at the age of seven or eight and graduates them at nineteen prepared either for the university or else for a profession or business life.[3] Since 1837 preparatory schools have fitted boys for the higher classes of the Public and Grammar Schools and for the Royal Navy. They keep boys to the age

[1] Cardinaal, *op. cit.*, pp. 336-337.

[2] Sharpless, *English Education*, p. 89.

[3] *Ibid.*, p. 81

of fourteen, but no longer.[1] Geometry is taught in the upper classes of these schools as well as in the secondary schools themselves. We are concerned also with another class of secondary schools, the technical institutions. The University of London, which was founded in 1836, was in part an outgrowth of a demand for technical and industrial education. Various technical and industrial schools were also created about this time, some of which have been preparatory to the University of London and the other modern universities since founded, while other technical schools of a high rank are affiliated with the universities and offer certain equivalent courses. Some of the technical schools are classed with the Grammar Schools. In fact, it is hard to draw any fast line between the various kinds of secondary schools in England.[2] Geometry of a practical nature is also taught in the upper classes of some of the elementary schools. In the higher elementary schools, where the pupil must be not younger than twelve, and remains three years, more serious attention is given to the subject.

We now turn to the teaching of geometry in the above-mentioned schools. It appears that in some of the higher elementary schools both Euclid and "geometry" are taught, the latter term, as the syllabi of instruction indicate, referring to practical geometry. In one of the typical London schools[3] the two following parallel courses are given: In the first year,[4] easy problems on plane figures. In the second year, Euclid I (propositions 1–15), with easy deductions; definitions, postulates, and axioms. What is called "geometry" is taught at a separate hour, it pertaining to the measurement of the triangle, circle, and regular polygons. In the third year, Euclid I with easy deductions. Also at a separate hour elementary "geometry" of the plane, and of the ordinary solids "in easy positions," and simple sections. We thus see that the "geometry" work is becoming a course in mechanical drawing. In the fourth year[5] (course

[1] Cotterill, *Preparatory Schools for Boys.* In Special Reports on Educational Subjects, Vol. 6, 1900, p. 1.

[2] The "Public Schools Year Book" gives in its list schools that offer quite technical courses.

[3] Medburn Street Elementary School for Boys, *Syllabus of Instruction*, 1903-04.

[4] Algebra, arithmetic, and mensuration are also given this year.

[5] Trigonometry is also begun as a separate subject.

A), Euclid I–IV with deductions. The work in elementary plane and solid "geometry" is continued. In course B of this year, Euclid I is finished and also eight propositions of Book II. The work in "geometry" includes plans and elevations of regular solids in easy positions, together with plans and elevations of points, lines, and plane surfaces. We thus see in the above that the study of Euclid and the study of practical geometry extend through the four years, but as distinct courses. In the William Street High Grade School (Boys), in London, Euclid I with deductions is studied.[1] Book II of Euclid is studied by those students who wish to try the Oxford Local Examinations. There is the same course in practical geometry leading into mechanical drawing as in the school described above.

In the preparatory schools the pupils begin to study geometry at the age of ten or eleven,[2] but the work is not very extensive, two hours a week being devoted to Euclid.[3] "Experience shows that in a first term, a class of six or eight boys can easily learn thoroughly six or eight propositions as well as the definitions, axioms, and postulates." As a large number of "riders" are also taken, the progress through the book is not rapid. Up to the time of entering the secondary schools, the majority of pupils cover about three books of Euclid, while the best in the class may include Books IV and VI in the same time.[4] But the schools vary in this respect. At the St. Paul's Preparatory School in London the boy at fourteen goes to the Higher School, having done only Euclid I.[5]

The teaching of geometry in the secondary schools will now be considered. It was not until after the middle of the last century that Oxford and Cambridge began to influence the teaching of geometry in the secondary schools through their local examination systems. Previous to this time and after about 1829, the schools were teaching Euclid in varying degree, which shows that standards were being raised at the universities. The influence of Cambridge, in particular, must have stimulated the teaching of

[1] *Syllabus of Instruction*, 1903-04.

[2] See *Special Reports on Educational Subjects*, Vol. 6, 1900, p. 63, article by G. G. Robinson.

[3] *Ibid.*, pp. 249, 251, article by C. G. Allum.

[4] C. G. Allum, *op. cit.*, p. 255.

[5] Sharpless, *op. cit.*, p. 145.

geometry in the schools by her own system of honor examinations. Before the establishment of the classical tripos in 1824,[1] the university student worked his way to honors by successfully leading in the disputations. From 1824 to 1850[2] the classical tripos was open only to those who had already taken honors in mathematics, and hence it was known as the mathematical tripos, "the instrument by which the proficiency of students in mathematics came ultimately to be tested."[3] These honor examinations in the universities naturally stimulated prospective students in the lower schools to do advanced work, and so we find the study of Euclid being introduced in the secondary schools. We have already seen[4] that the St. Domingo-House School in 1839 provided for the study of Euclid I–VI in Latin for those intending to enter the universities.

In 1858[5] Oxford and Cambridge instituted their systems of Local Examinations, which were held at various centers throughout the Kingdom. Those who passed these were excused from the first of a series of three examinations leading to the B. A. degree in the universities. A need of a more thorough system of examining the schools was felt, so in 1873 was established the system of examinations by the Joint Board[6] of the two universities. These two systems of examining the schools obtain to-day. The universities as a whole do not impose an entrance examination, the various colleges of these institutions having separate standards. By 1873 only two colleges in Cambridge had entrance examinations or "required more than a certificate of fitness from any M.A. of Cambridge or Oxford." Tests were common at Oxford then, but the standards varied with the different colleges.

The schools listed in the Public Schools Year Book[8] generally offer two courses, the classical and the modern "sides." The commercial side also is being introduced, and the recent "Reg-

[1] Ball, *A History of Mathematics at Cambridge*, p. 211.

[2] Balfour, *The Educational Systems of Great Britain and Ireland*, p. 231.

[3] Ball., *op. cit.*, p. 187

[4] See above, p. 89.

[5] Balfour, *op. cit.*, p. 179.

[6] *Ibid.*, p. 182.

[7] *Ibid.*, p. 237.

[8] See statement of courses in the *Public Schools Year Book*, London, 1906.

ulations for Secondary Schools" recognize three types of courses, the classical, the modern, and the commercial, each type with a leaving age of nineteen, seventeen, and sixteen, respectively; or nine courses to select from.[1] One can see that the schools are beginning to break away from the traditional classics. The older Public Schools, however, still emphasize these studies.

The entrance requirements to the "Upper School" vary with the different institutions. They vary from one book of Euclid to three books. Thus Haileybury College recommends that the boy, if he has done any Euclid, be questioned upon it.[2] This shows no fixed standard. At Harrow, where the boys are admitted between the ages of twelve and fourteen, no geometry is required for those entering on the classical "side," but for the modern "side," mention is made of examining on the earlier part of Euclid or some other elementary geometry.[3] Here we observe a tendency not to hold to Euclid for admission.

When the English boy is thirteen years of age he has studied from one to three books of Euclid. Bright boys have studied more, since according to the English custom the brighter students are allowed to advance as fast as possible. The plan of giving scholarships and other prizes helps to bring this about. When the boy has completed the course he has usually finished Euclid VI and XI. If he is specializing in mathematics for the university scholarships, he studies during the Sixth Form higher branches of mathematics.

The technical secondary schools of London, which are in close relation with the University of London, generally have both day and evening classes. The course in mathematics in some of the schools prepares students for the annual examinations held by the Board of Education and leads toward the B.Sc. degree in the university. The City of London College[4] offers the following course in elementary geometry: *Stage* 1. Measurement of angles, geometry of similar figures, mensuration of the triangle and circle. *Stage* 2. Simple problems in heights and distances

[1] Brereton, *A Comparison between English and French Secondary Schools.* In the *Journal of Education*, London, Vol. XXVII, 1905, new series, p. 123.

[2] The boy is about thirteen years of age. See *Public Schools Year Book*, 1899, p. 123.

[3] *Public Schools Year Book*, 1899, p. 131

[4] Calendar for 1903-04.

of inaccessible objects. Geometry of the plane. Mensuration of the three round bodies. Elementary spherical geometry. *Stage* 3. Ratio and proportion with application to geometry as far as the subject is treated in the definitions of Euclid V and VI. For the work in these three stages, Hall and Stevens' Euclid is recommended. In the Woolwich Polytechnic[1] the elementary course in geometry comprises the above three "stages" and also a fourth, which includes the study of Euclid XI, the prism, pyramid, and the three round bodies. In both of these schools there are also courses in practical plane and solid geometry and in advanced mathematics. The Sir John Cass Technical Institute[2] offers practically the same work, but more particular reference is made to the study of Euclid in the various "stages." Thus the substance of Euclid I with exercises is required for stage 1, and the substance of Euclid II, III, and IV with exercises for stage 2. The work required in these two "stages" fulfills the matriculation requirements in elementary geometry at the University of London.[3] The university calendar for 1904–05 makes this requirement: "The subjects of Euclid I–IV with simple deductions including easy loci and the areas of triangles and parallelograms of which the bases and altitudes are given commensurable lengths (Euclid's proofs will not be insisted on)."[4] The questions on the above subject-matter relate both to the practical and the logical. One of the questions illustrates the combination of the two: "The sides of a triangle are 2.6 in., 2.4 in., and 1 in. State and prove the proposition by which it may be shown that one angle of this triangle is a right angle."[5]

The geometry taught in these polytechnic schools, while of a practical nature, still shows a tendency to stand by Euclid. One gains the impression from the calendars that it is not the custom in many of the schools to precede the logical geometry

[1] *Syllabus of Classes*, 1903-04.

[2] *Syllabus of Classes*, 1903-04.

[3] *Calendar*, 1904-05, Vol. I, p. 4, and appendix.

[4] In 1845, Euclid I was required. The questions were elementary Fowler, *The Teaching of Mathematics*; in *Essays on Secondary Education*, edited by Christopher Cookson, Oxford, 1898, p. 237.

[5] *Calendar*, 1904-05, Vol. I, appendix, p. vii. The questions were set in Sept., 1903.

by practical exercises. The substance of Euclid I with exercises usually constitutes the subject-matter of stage 1.

One would not expect the Public Schools of England to attempt any reform in the teaching of geometry. They are bound too closely to the two great universities through a rigid examination system which insists on a close adherence to Euclid. These schools for the most part prepare for the "higher callings," and any taint of commercialism would not be consistent with their traditions. Hence practical mathematics is not in favor, and Euclid still holds sway. Reform may be looked for from the other Grammar Schools that tend to emphasize the "modern side." The secondary schools of London, many of which offer technical courses, stand in a position to make radical departures from tradition. That they are not adhering to Euclid, we have already seen.

In 1870 the Association for the Improvement of Geometrical Teaching was organized, and its syllabus, which appeared in 1875, sought to make a departure from Euclid and thereby improve the teaching of geometry. This syllabus departed from the sequence of Euclid and introduced changes in the subject-matter, particularly in the treatment of proportion, but little improvement has resulted in the teaching if we are to judge by the recent reform movement of which we shall now speak.

At the Glasgow meeting (1901) of the British Association, Professor John Perry read a paper[1] on the teaching of mathematics which suggests some aggressive reforms. He deplores the slavish adherence to Euclid and urges that mathematics be made practical, that it be closely correlated with science, and that the student be permitted to get early into the rich field of higher mathematics by assuming a large part of elementary geometry. Concerning this last point, Professor Perry says: "Where would be the harm in letting a boy assume the truth of many propositions of the first four books of Euclid, letting him accept their truth partly by faith, partly by trial? Giving him the whole fifth book of Euclid by simple algebra? Letting him assume the sixth book to be axiomatic? Letting him, in fact, begin his severer studies where he is now in the habit of leaving off? . . . We shall thus get the same intellectual training with

[1] Printed under the title *Discussions on the Teaching of Mathematics*, London, 1901.

more knowledge."[1] In the discussion following the reading of Professor Perry's paper several speakers referred to the evils of the present examination system. In reply, Professor Perry heartily concurs and adds, "I assert that we want reform of a whole system of dunce manufacture of which examination is only one part."[2]

Two things above all stand out in Professor Perry's recommendations: (1) That much of elementary geometry be assumed as axiomatic, and (2) that the subject-matter be taught with reference to its utility. We get a better idea of his suggested reforms by examining his syllabus,[3] which has been prepared for use in the training colleges.[4] Under the topic mensuration is included experimental work, such as testing the rules for the areas of the ordinary plane figures, the ellipse, the surface of a cone, of a cylinder, by means of scales and squared paper. Propositions in Euclid are also tested in the same way. Professor Perry recommends as a course in geometry: "Dividing lines into parts in given proportions, and other experimental illustrations of the sixth book of Euclid. Measurement of angles in degrees and radians. The definitions of sine, cosine, and tangent of an angle; determination of their values by graphical methods; setting out of angles by means of a protractor when they are given in degrees and radians, also when the value of the sine, cosine, or tangent is given. Use of tables of sines, cosines, and tangents. The solution of a right-angled triangle by calculation and by drawing to scale. The construction of any triangle from given data; determination of the area of a triangle. The more important propositions of Euclid may be illustrated by actual drawing; if the proposition is about angles, these may be measured by means of a protractor; or if it refers to the equality of lines, areas or ratios, lengths may be measured by a scale, and the necessary calculations made arithmetically. This combination of drawing and arithmetical calculation may be freely used to illustrate the truth of a proposition. A good teacher will occasionally introduce demonstrative proof as well as mere measure-

[1] Perry, *op. cit.*, pp. 12, 13.

[2] *Ibid.*, p. 93.

[3] *Ibid.*, pp. 25-32.

[4] These train teachers for the elementary schools

ment."[1] Of those who took part in the discussion at the Glasgow meeting the majority agreed with Professor Perry in his attempted reform. The opinion of W. D. Eggar of Eton College shows the attitude of a representative of one of the great Public Schools. He thinks the syllabus excellent, but says there would be great difficulty in adapting it on account of a lack of fully qualified teachers. Mr. Eggar suggests the following treatment of geometry: "A course of geometrical drawing to be agreed upon which should replace Euclid II, IV, and VI. Books I and III to be taught still, but in conjunction with geometric drawing. Euclid II (12, 13) to be transferred to trigonometry, which should begin at an earlier stage."[2] Professor Perry's reform has caused much discussion in England and America. Many English teachers of mathematics are heartily in favor of a change, but they recognize the difficulties in the way, the greatest of which is the rigid examination system controlled by the universities.

The "Perry Movement" will be referred to again in another connection.

OTHER EUROPEAN COUNTRIES

The Gymnasia of Sweden were established in 1623 and trained almost exclusively for the clergy and civil offices.[3] These schools to-day are offering the classical and modern courses, covering nine years' work. Mathematics receives its full share of attention. Geometry is usually begun in the second year class and continues with the rest of the mathematics through six years. More attention seems to be given to Euclid than in Germany, but great freedom is allowed the teachers in a choice of texts.[4]

[1] Some of the recommendations suggested by Professor Perry are by no means new. In an article in the *Quarterly Journal of Education* (London), 1833, it is advised that there should be a preparation for scientific geometry by means of discovering truths from accurately drawn figures. The propositions of Euclid are to be tested by means of instruments and by means of paper cutting. See *Quarterly Journal of Education*, 1833, Vol VI, pp. 35-49; 237-251.

[2] Perry, *op. cit.*, p. 81.

[3] *Sweden, its People and its Industry*, edited by Gustav Sundbarg, containing an article on secondary education, pp. 33-49.

[4] See the *Redogörelse* for each of the secondary schools at Malmo, Lund, Linköping, Kalmar, Kristianstad, and Karlstad; 1894-95.

The present organization of the Gymnasia of Norway[1] dates from 1869, when the literary and scientific courses of six years each were instituted. The middle schools prepare for the Gymnasia and for practical life. Geometry is begun in the middle schools in a practical way and is continued in the Gymnasia, where it is taught in a systematic manner.

The Latin and the Realskoler of Denmark,[2] which received their present organization in 1850 and 1871, correspond to the Gymnasia and the Realschulen of Germany. Geometry is taught throughout the six years' course of the Latin schools, where the pupils are prepared for the higher professions and for the university.

Although the teaching of mathematics has had a late development in the secondary schools of Austria,[3] since 1884 this subject has ranked equally with the languages and history. The institutions which teach elementary mathematics correspond to those of Germany.

Bulgaria also has the two main types of secondary schools, the Gymnasium and the Realgymnasium. In the four higher classes of each, both algebra and geometry are taught.[4]

One gains a good idea of the teaching of mathematics in the secondary schools of Switzerland after understanding the aims and methods of the German institutions.

Belgium[5] also has its secondary institutions similar to those of Germany. Mathematics ranks equally with the other subjects.

The secondary schools of Spain[6] offer three courses, each extending over five years. Practical geometry and arithmetic are taught in the first year. The teaching of geometry becomes more systematic in the third year, when trigonometry is introduced. Algebra is begun in the second year.

[1] Baumeister, *op. cit.*, I_2, article on Norway, pp. 399-401; Hippeau, *L'instruction publique dans les états du nord*, pp. 180-183.

[2] Baumeister, *op. cit.*, I_2, article on Denmark, pp. 385-388; Thornton, *Recent Educational Progress in Denmark*, in Special Reports on Educational Subjects (London), 1896-97, pp. 587-614; Hippeau, *L'instruction publique dans les états du nord*, pp. 233-244.

[3] Beer und Hochegger, *op. cit.*, Vol. I, pp. 266-532; Simon, Ens. Math., 1902, pp. 157-166.

[4] Rein, *op. cit.*, I, article on Bulgaria, p. 825.

[5] *Ibid.*, article on Belgium, pp. 464-466.

Baumeister, *op. cit.*, I_2, article on Spain, pp. 725, 727.

THE UNITED STATES

In the United States, elementary geometry is taught principally in the high schools. It is also found in the courses of the normal schools and the few academies. In the normal schools, the work varies greatly. Some give courses in plane geometry, others in both plane and solid, and still others require plane geometry as a requirement for admission. There are also various kinds of polytechnic schools where geometry is taught more or less with respect to its applications. Our grammar schools now generally give some attention to practical geometry either in connection with mensuration as a branch of arithmetic or as a separate course, where the pupil is taught simple constructions and learns the properties of the common geometric figures.[1]

Instead of having separate institutions for the various courses, as is the case in Germany, the American high school offers from two to four courses. The Committee of Ten[2] (1892) recommended the following four programs: Classical, Latin-scientific, Modern Language, and English. In all these, geometry begins in the second year of the four years' course. Algebra is begun the first year, when the pupil is about fourteen years of age. Both studies continue through the third year, when geometry is completed, including both plane and solid. In the fourth year trigonometry and higher algebra are required in the English course. In the other courses there is a choice between trigonometry and higher algebra or history. Although the work of the Committee of Ten has influenced the curricula of the high schools, it has by no means brought about uniformity of courses, a situation perhaps not desirable in this country. The recommendations of the committee, however, will give one a fair idea of the curriculum of the average American high school.

Above all, the high school geometry is logical, although Euclid as such finds little favor with us. The work is generally begun in the second year. Two years earlier, in the elementary school, the student has had some form of concrete geometry. There is no bridging over this gap, it being only the occasional teacher who prefaces the high-school work by some form of inventional geometry.

[1] This was recommended by the Committee of Ten, appointed by the National Educational Association. See *United States Bureau of Education*; *Report of the Committee on Secondary School Studies*, 1892, p. 23ff.

[2] *Op. cit.*, p. 23ff.

The conduct of the class is quite different from that in Germany or in France. In Germany, for instance, the class method predominates. In the United States, the individual is the center of interest. This is seen in the arrangement of blackboards. One black-board suffices for the German or French schoolroom where the class as a unit develops a particular theorem. With us it is necessary to have a continuous board around the room, so that each pupil can draw his own particular figure from which to explain. The above typifies what has been, and is yet to a large extent, our common schoolroom practice. Nevertheless we have teachers who follow the German plan and still others who try to combine these two extremes. The dependence upon a text-book and the requirement of a considerable amount of home work prevent the class period from being a time for instruction. Hence our teachers in mathematics tend too much to "hear" recitations. We have much to learn from Germany and France in this respect.[1]

Since 1871, when the University of Michigan established the system of accrediting high schools, the teaching of mathematics in the latter schools has been more largely influenced by the universities. We find teachers, therefore, more busily engaged in preparing students for the university than for their daily living. The large universities of the east have not generally adopted the accrediting system, several admitting only by examination. Hence there have been varying standards of admission. Uniform entrance requirements being desirable, the College Entrance Examination Board, made up of representatives of the various universities concerned, was organized in 1900. This will undoubtedly secure greater uniformity of work in the various preparatory schools and will also improve the teaching of geometry from the point of view of the university, but it does not necessarily follow that this will be best for the pupils.

Within the last few years a movement of another sort has gained some prominence in America. Professor Moore of the University of Chicago was the first prominent mathematician in this country to champion the cause of the "Perry Movement."[2]

[1] Chapter VII below contains much that pertains to the teaching of geometry in our schools.

[2] See above, pp. 126-128.

This position was taken by him in his presidential address[1] before the American Mathematical Society, in December 1902. The result has been that in the Middle West many of the highs chools in touch with the University of Chicago have begun a reform in the teaching of mathematics. They seek to teach the various mathematical branches in close correlation, this to be brought about by making mathematics the tool for scientific investigation. We thus see the tendency. Geometry, for example, is not to be taught solely for its logical value. What can be done with it is the chief desideratum. As a result of the interest aroused by this new movement, five associations for teachers of elementary mathematics were organized in 1903 alone.[2]

The Central Association of Science and Mathematics Teachers presented three objects in its organization:[3] 1. To promote better teaching of science and mathematics, especially in the secondary schools. 2. To obtain a better correlation of these subjects to each other and to the other subjects of the curriculum. 3. To bring the college and the secondary school into closer relations. The second of these objects indicates the keynote of the new movement already mentioned.

[1] Moore, *On the Foundations of Mathematics*, Bulletin A. M. S., 1903, p. 424; also School Review, Chicago, Vol. 11, 1903.

[2] The Central Association of Science and Mathematics Teachers, with headquarters at Chicago; the Association of the Teachers of Mathematics in the Middle States and Maryland; the Association of Mathematical Teachers in New England; the Association of Teachers of Mathematics in Washington (State); and the Association of the Teachers of Mathematics in Kansas. Since 1903 other States have organized similar associations.

[3] This association stands for the "Chicago Movement."

CHAPTER VII

PRESENT PROBLEMS AND THEIR HISTORIC CONNECTONS

AN HISTORIC SURVEY

Certain features have characterized the development of geometry and its teaching that have a particular bearing on present problems and practices. In its geometric development, the race has passed through at least four well-defined stages. There was first the intuitive stage, in which geometric forms and principles were adapted to the needs of daily living, without, however, any principle of classification or generalization. A second stage was marked by an ability to recognize space forms *per se*, and to classify geometric notions sufficiently to be able to deduce and apply adequate, although not always exact, working rules. A third stage was reached when geometric relations were systematized into a logical sequence. Lastly, practice was again developed, this time controlled by logic.

In this order of development, it is to be observed that:

1. *The practical*[1] *preceded the logical.*—The Egyptians, Babylonians, and Chinese had a knowledge of practical geometry before Greece invented its logic. Before the study of Euclid gained a foothold in the schools of Europe, the practical geometry of the Romans had been developed. With the Hindus and the Arabs, the practical was of prime importance even after the latter became familiar with the Greek learning. In later times, the various countries of continental Europe paid more or less attention to the practical, and Russia in particular was late in assigning a prominant place to logical geometry.

[1] The term practical is used as heretofore with reference to the applications of geometry within the field of mathematics or in the related fields of science.

2. *The transition into the logical was not abrupt.*—Thales based his deductive geometry on the practical work of the Egyptians. He and his school were interested in astronomy, and to them are assigned some practical problems in geometry. The few theorems assigned to Thales show what a small beginning was made in logical geometry at Miletus. Pythagoras was the first to sever geometry from the needs of practical life and make it an abstract science. The first Greeks who developed a logical geometry did not make, then, an abrupt transition into it, but were guided in part by practical considerations.

When geometry began to be taught in the medieval universities, it was confined to the learning of definitions with perhaps some practical applications. The logical work was begun with the teaching of Euclid. The early secondary schools which have been previously mentioned gradually introduced the logical, the first geometry being taught in connection with astronomy or geography. It seems to have been generally recognized that the logic of Euclid was difficult.

We thus see that the first development of logical geometry was not completely divorced from practical applications, and, in a greater or less degree, its first teaching in the early universities[1] and secondary schools was not according to the "Elements."

3. *The tendency to hold to the special has been marked.*—We have seen that even the Greeks in their logical work found difficulties in proving a general theorem. Euclid, in some of his definitions, fails to recognize the concept of the general. The Italian practical geometries of later times illustrate this tendency to a marked degree. These books show but little the influence of the "Elements," and so illustrate what may be called natural tendencies.

4. *Geometry has been taught more and more to younger students.*—With the Greeks, the study of geometry was for mature minds. Plato, we recall, believed it should be studied between the years of twenty and thirty. The medieval universities taught Euclid in the higher classes. When our own universities were founded they taught geometry in the last year. In later times it was taught in the freshman year, and finally the high schools took up the work. Geometry is now taught in our high schools generally in the second year, when the pupil is about the age of fifteen.

[1] This was for a short period only in the universities.

In France and Germany the boy is doing logical work at least three years before this. With such a changed position of geometry in the course of study, one should expect to find a change of aim and method.

In this historic survey, we are also concerned with:

1. *The sequence in the subject-matter of geometry.*—Euclid did not follow the historic development in his sequence of subject-matter, for the practical was entirely eliminated, and the Greeks began the development of the subject-matter of geometry in quite a different order from that found in the "Elements." The Pythagoreans studied the regular solids before plane geometry had received any great development, and they also studied proportion, which comes late in the "Elements." Definitions and axioms, which introduce the first book of Euclid, were given prominence by Plato and Aristotle about 200 years after Thales founded the science. The sequence in three of Euclid's definitions is not pedagogical. He defines point, line, and surface in the order named, placing first the most remote from experience. This order has prevailed to a large extent up to recent years. One marked exception is found in the geometry of Gerbert, where the above order is reversed, the definition of a solid being placed first. Modern practice adopts the order of Gerbert.

2. *The sequence in the development of the mathematical subjects.*—Historically, arithmetic, geometry, trigonometry, and algebra have been developed in the order named, but not in a strict "tandem" order, for there has been considerable overlapping. Elementary geometry alone was systematized within 300 years, but even it, within a century past, has annexed a vast field, including, among other topics, the geometry of the triangle and circle, the theory of transversals, and radical axes.

3. *The correlation of geometry with science.*—Thales was an astronomer and employed some of his theorems in practice. One cannot say positively that there was any correlation between mathematics and science in this first development of the logic of geometry, but there can be little doubt that these early investigators were stimulated to further study of geometry through their interest in astronomy. The Athenian Greeks were interested in both science and geometry, but the former was divorced from the latter. The same relation existed between

these subjects in the early universities and also in the universities of later times. In the early history of the secondary schools of the countries considered above, excepting England, geometry was taught in some degree in connection with surveying and astronomy, but in the nineteenth century, with its specializing tendencies, the teaching of the subject became practically independent of any such associations. Thus down through the ages to modern times, while geometry has found its applications in the fields of science, the teaching of geometry has, on the whole, been independent of such applications.

4. *The question of method.*—The various methods of attack in common use in the study of geometry to-day were developed by the Greeks. Of these we recall the method of exhaustion, reductio ad absurdum, analysis, and the solution of problems by the intersection of loci. The effectiveness of the last method has been fully recognized, however, only in modern times. The method of reduction as developed by Hippocrates is in common use to-day, although not usually defined. The value of the inductive method is not yet fully understood. In a wider sense, we are indebted to the Greeks for a method of instruction, the Socratic, or that of question and answer.

Educational aims have an influence on method. With the early Greeks the science of geometry was being developed. Students were taught in part that they might do something to add to this development. After Euclid, geometry was considered finished. It was taught so that students might have power in learning what had already been perfected. This naturally led to learning by heart. In the universities, as well as in the secondary schools from the early times down to the present, students have studied geometry largely for its disciplinary value. Form has been studied to the exclusion of thought. In recent years it is beginning to be recognized that the mere learning of theorems will not give greater power of reasoning. Original work now occupies a prominent place in the teaching of elementary geometry.

With the Greeks, individual instruction seems to have been a characteristic feature in the teaching of geometry. In the universities, where the lecture plan prevailed, the individual was lost sight of. In Germany and France to-day the class is the unit for instructon. In England, it is the individual, where, by

scholarships and other prizes, each student strives to outdo his fellow. In our own country we are between these two extremes, but lean toward the development of the individual.

In answer to the question why geometry has been taught, two reasons may be mentioned. It has been taught for its practical and logical values, and of these the logical has stood out the most prominently. Before the sciences of the present day entered the schools, the classical and mathematical studies afforded the supposed requisite mental discipline. And it was and still is to a great extent believed that this training gives to the student a certain formal power which he can apply in other fields, and our methods have been shaped accordingly.

THE EDUCATIONAL SITUATION TO-DAY

We are living in an age which demands action. It is no longer sufficient that one be able to think logically, but one must be able to *do* things as well. The demands of society are so complex that the finishing schools must train along many lines and not, as in the days past, train simply for the "higher callings." So we have many special schools that train for various vocations. In Germany the public schools seem well adapted to meet the present conditions. There, the boy, if he must earn a livelihood as soon as possible, attends the elementary school (Volkschule), or at most the high grade elementary school, or perhaps the Realschule. The Realgymnasium and Oberrealschule prepare for the commercial and industrial callings, and the Gymnasium for the "higher callings." In England, also, there are schools that provide for class distinction, but the secondary schools, for the most part, prepare for the universities and the positions to which gentlemen of quality are called. The rise of technical schools in England is in response to a demand for better training for effective service in industrial and technical lines. In the United States we have a few high grade technical schools, but we are behind England in the organization of secondary technical education. Unlike Prussia, our public school system is not well adapted to meet the various demands that are put upon the schools. We have one set of institutions, through which must pass pupils with different aims in life. When the elementary school is finished, one pupil begins to function in the world, another enters the high school.

In Germany, two such boys would be in different institutions from the beginning. In our high school opportunity is given for a selection of courses. In Germany, the boy of high-school age has a choice of several institutions. While the American high schools are beginning to offer certain courses, for example the commercial, that prepare for immediate life work, the pressure of the university is so great that the high school finds it difficult to perform the double function of serving both as a preparatory and as a finishing school.

The new education is demanding a preparation for effective service, whether this be in the elementary school, the secondary school, or in the university. The school subjects should not be taught merely as a preparation for higher studies in the same or kindred lines. They should have a present value for the student. This requires a closer unity between schoolroom practice and the life the pupil is then living, and the problem as it relates to geometry will now be considered.

PRESENT PROBLEMS IN THE TEACHING OF ELEMENTARY GEOMETRY

The teaching of geometry has been concerned primarily with its subject-matter and its logic. There has been the field of geometry into which the learner enters. The aim has been to acquaint him with this field. In learning its content according to prescribed rules, the mind of the student is trained in logic. The student has been adapted to set standards. One thing that has not been done is to recognize that each student has an individual psychology and that, to meet these conditions, geometry should lose its traditional fixedness and be made adaptive to the needs of individuals. Good teaching will take cognizance of three factors involved. The first consideration is the psychology of the individual; the second, the need of the logical training of the mind; the third, the recognition of the demands of the present age. The history of the teaching of geometry gives us a perspective whereby we can the better give attention to these considerations. It has also a prospective side in helping us to guard against the errors of the past and points out that which has been of value. Certain present problems in the teaching of geometry have a relation to the foregoing historic material. We shall here be concerned with:

1. *The balance between the logical and the practical.*—Logical geometry is generally associated with the theoretic. The history of the teaching of geometry points this out. Logic did not enter the practical work of the Egyptians, while the Greeks on the other hand did not admit practice in the theory. In later times, we have seen that practice has entered into the school work, but at most it has followed the theory, not served as an introduction to it. Thus logic has been disassociated from practice. When theory, however, is approached inductively, opportunities are offered for practical logical work. We are so accustomed to use in geometry the term logic for deductive logic that we have almost forgotten that inductive logic has a place in teaching. From a series of concrete experiences, the pupil may be led, after the inductive method, to formulate some general principle. And the work can be made practical by employing the scale, the protractor, the compasses, and instruments for field work, not to mention the related numerical exercises. This inductive work forms a fitting propædeutic to the ordinary high school course in geometry, and finds a fitting analogy in the historyof geometry where the practical stage preceded the logical. Even after deductive geometry is begun, the inductive treatment could well be continued as a method of discovery.

Practical field problems serve to point out the existence of certain theorems, which should then be logically proved. We thus see the need of a continual co-operation between theory and practice. The latter serves to verify the former and many times to check erroneous conclusions. This is seen in the practice of drawing accurate figures. Such a plan serves not only to check illogical conclusions, but frequently to suggest a line of proof by mere observation of the relations between the lines or angles drawn.

The history of the teaching of geometry tells us that the mind tends to hold to the special and approaches the general with effort. We have seen this in the Greek development and especially in the practical geometries of later times. We should give heed to this tendency in our teaching and not try to force too early the idea of the general. This argues here again for the inductive method in the first stages of the work. As a result of following to the extreme the deductive method, we find our geometries stating the general theorem, followed by the proof

and a list of corollaries. At least some of these could well be used as special cases which lead up to the general theorem. This of course does not apply where the general theorem is of a very simple nature.

Our history also tells us that many of the geometries in use in the past have employed fallacious proofs. This has already been pointed out in certain French and German texts. Perhaps the most common example of this was in connection with the theory of parallels. Some of the simpler theorems were sometimes stated without proof. Has this any significance for us in our schoolroom practice to-day? Are we to require all students to master the most difficult propositions? Are we to assume any theorems without proving them, as Professor Perry suggests? It would be impossible to answer these questions in the light of history. That such practices were common is known. France and Germany have produced more famous mathematicians than has England, which has closely adhered to the logic of Euclid. But if France and Germany had adhered closely to Euclid, would they have produced still a greater number of famous mathematicians? The question must remain unanswered. If the teacher desires to defer a certain proof on account of its difficulty, or to take for granted some theorems seemingly self-evident, he certainly is following historic precedent. It seems wise to assume, at least tentatively, some of the most simple propositions. For the beginner, these are the hardest to understand. The theorem that all right angles are equal was taken as axiomatic by Euclid, but modern books give it as a theorem. It is capable of proof, but when proved with all the "machinery" that accompanies a proof, it becomes difficult for the beginners. It is better to test the truth of the statement by paper-folding, and then set up the theorem as one to be quoted. As an example of theorems that could well be deferred, all teachers recognize the difficulties in presenting the incommensurable case in the applications of the theory of proportion to various magnitudes. The texts used in France and Germany up to the nineteenth century, and perhaps later, usually neglected this case entirely. There may be a question as to what extent texts can be arranged to accommodate these breaks in logic, but the teacher does not depart from precedent if he defers any theorem or group of theorems for later treatment.

Practice left unguided by theory is apt to lead to erroneous results. The Egyptians used incorrect formulas for certain rectilinear areas. Students of geometry to-day often adopt these practices. It is the province of theory to show wherein the error lies and point to the correct solution. But theory often takes the concreteness out of things. The fact that a boy uses an erroneous formula of his own construction does not by any means indicate a low order of thinking. The great Gerbert, who knew little of Euclid, found incorrectly the area of an isosceles triangle. The boy who has erred as described has used his judgment, which is not always the result of a training in logic. In rectifying such errors the pedagogic teacher will seek to stimulate the right growth of such judgment and not replace it by a blind adherence to the results obtained by logical reasoning.

In seeking a balance between the logical and the practical, we may observe that:

a. The general should be approached through the special in early work. Practical tests have a place here in serving to check theory. Such work may lead to discoveries.

b. Theory is a guide to practice.

c. Logic need not be divorced from practice. Inductive geometry has a practical aspect.

d. Induction should be employed even after the deductive study is begun.

e. There is historic precedent in assuming self-evident theorems and such as require a very high grade of reasoning.

2. *The sequence in the subject-matter of geometry.*—A sequence that is logically best is not necessarily pedagogically the best. The logical sequence is a thing that concerns the science of geometry. It is a passive, immovable thing. The pedagogic sequence—if we could have one—concerns the minds and aptitudes of the pupils and the demands of present living. It must be alive and subject to change. So a geometry constructed to meet the needs and demands of the student must necessarily be different from one that seeks alone to perfect a system of logic. Teachers of mathematics above all others have been slow in recognizing these distinctions. Text-books, beginning with Euclid, have, to a large extent, been written from the standpoint of the development of the science rather than from that of the needs of the pupil.

One of the bad features of following a prescribed text is that it is so easy to follow it and thereby to neglect the needs of the class. The class hour is usually employed in working over theorems already explained in the text or set down as originals. It should be a period for investigation and discovery. Under such a plan two classes could not follow the same sequence nor work the same theorems and exercises. What is needed is more independence on the part of the teacher. If the needs of the class demand it, he should not hesitate, for example, to bring down into the early work features usually deferred, such as the notion of locus, and problems involving the intersection of loci.

The sequence in Euclid's "Elements" does not correspond to the order in which the subject-matter was worked out by the earlier Greeks.[1] The recognition of this fact should influence teachers in their class work. The Greeks were impelled to further study in the hope of discovering new truths. Since the time of Euclid very little has been added to the subject-matter of our modern texts that is not closely related to the lines laid down by Euclid. Teachers have taught geometry under the impression that power is gained by thinking in the same channels as Euclid did in compiling his "Elements." They have lost sight of the fact that the real thinkers were those who made this subject-matter accessible for Euclid. What we need then is to train our students to be investigators, practically and logically. Holding to a fixed sequence of propositions founded on logic alone will not bring this about.

The sequence that places solid geometry after plane is not according to the historic development, for solid geometry was begun before plane geometry was fully developed. In their investigations the Greeks did not think it necessary to wait until the geometry of two dimensions was fully developed before passing into the one next higher. But that is what we have done since Euclid set us the example. Whether plane geometry and solid geometry can be taught simultaneously with success is not yet fully determined. The plan is now being experimented on in France and in Italy, and good reports come of the progress made. The principle of analogy is the basis for this correlation. Thus when the pupil is studying the plane angle, the idea of the dihedral angle is introduced. The area

[1] See above, pp. 16, 29.

of a triangle has a correspondence in the volume of a tetrahedron. The treatment of similar figures in plane geometry bears an analogy to the same in solid geometry. This is particularly noticeable when these definitions are based on the notion of ratio of similitude.[1] The advocates of the method of "fusion" claim a great saving of time over the ordinary methods.

In choosing a sequence for the subject-matter of geometry, we should therefore remember:

a. The historic development does not correspond with that found in Euclid and later texts.

b. The logical is not necessarily the pedagogical.

c. Holding to a fixed sequence will not make students investigators in the best sense. The sequence should be subject to change.

d. The analogies between plane and solid geometry should be emphasized.

3. *The sequence of the different mathematical subjects.*—According to the historic development, plane and solid geometry preceded the study of algebra and of trigonometry, and followed that of arithmetic. There were national and natural reasons for this sequence. Arithmetic was developed first, for it was first needed in primitive industrial and commercial life. Practical geometry would naturally follow, coming before the logical. Algebra, more abstract than geometry, was developed later. Trigonometry was an outgrowth of geometry. Elementary geometry was practically unchanged after the Grecian period, but trigonometry and algebra in particular have received their development through succeeding centuries.

A common sequence to-day in our high schools is algebra in the first year, taught as an abstract science. In the second year, plane geometry on a strictly logical basis. Then follow some advanced algebra, solid geometry, and trigonometry, usually in the order named. Such a plan ignores what it seems wise to copy in the historic development and copies that which it should not. Since we teach mathematics for logical, as well as practical reasons, there is strong argument for beginning the high school course with geometry instead of algebra, if there is to be any "tandem" arrangement. On the other hand we have copied unwisely from history in placing trigonometry after solid geome-

[1] See Beman and Smith's *New Plane and Solid Geometry.* p. 182ff.

try. Beginning with Heron of Alexandria, the measuring of heights and distances, which involved a rudimentary trigonometry, was placed either in the midst of, or after the treatment of, solid geometry in many of the books written on practical geometry. Plane trigonometry involves no principles not found in plane geometry. When taught after solid geometry, if there is not a break in the logical order, there is at least a loss of time. We can learn a great deal from the French and Germans in this respect. The simple trigonometric concepts can be introduced while the student is studying proportion, if not before, and simple problems in the measuring of heights and distances can be solved by geometry and by trigonometry, the latter gradually superseding the former. Even if the regular study of trigonometry is not undertaken at this time, some of the practical phases can well be utilized to advantage.

While it is recommended that geometry be begun before algebra, the student should be familiar with the elements of algebra at least by the time proportion is reached in geometry. The Greeks employed the geometric theory of proportion and had no need for the algebraic, but to-day we assume the one-to-one correspondence between these two subjects, and with this convention, proportion in geometry is treated algebraically. In the treatment of proportion there is an opportunity for the teacher to relate arithmetic, algebra, trigonometry, and geometry. The fundamental theorems of proportion should not be proved algebraically, and then a few months later be applied to the geometric figures. Such a procedure causes a gap between theory and practice. Many students are prone to quote a series of theorems meaningless to them. When the theorem in proportion is proved, it should immediately be accompanied by a geometric and arithmetical illustration, or better, if the teacher is skillful, be applied immediately in some theorem or problem. This argues again for a flexible sequence of subject-matter. Algebra can be applied in some of the propositions in geometry that are based on proportion; for example, in the theorem, "If from without a circle a tangent and a secant be drawn, the tangent is a mean proportional between the whole secant and the external segment." Four line segments are here involved. Assign numerical values to three of the segments and require the finding of the value of the fourth. This requires the solution

either of a simple equation, or of two types of quadratic equations, according to which segment is taken as the unknown. After the answer is found, the student should compare the result with the length of the corresponding segment in the construction. By such a method as the above, algebra and geometry mutually assist each other. The checking of the result by measurement is not the least valuable part of the process The simultaneous teaching of algebra and geometry has been tried in this country and has met with some success. The chief objection to the plan has been that there is a lack of concentration of effort and hence a loss of power on the part of the pupil. This will undoubtedly be the case if the subjects are taught merely simultaneously and not correlatively. That such a plan is successfully carried out in France and in Germany has already been mentioned.[1] By such a plan the student of elementary mathematics gets a breadth and comprehension of the subject as a unit, which is entirely impossible when they are taught in "water-tight compartments."

The plan of teaching the various mathematical subjects in close correlation has the advantage of letting the student see the significance, the "worth-whileness" of what he is learning. It is in practice that the full meaning of what he is learning is brought out, and this correlation implies practice. But the full significance of the several branches in elementary mathematics is not realized by the student until they are taught in relation to other fields of science, in particular to physics and astronomy. Some of the schools under the influence of the University of Chicago have sought to teach mathematics in relation to the sciences. When it is suggested to teach the several mathematical branches correlatively, the question of shaping the curriculum for this purpose presents a problem, but when it is also suggested that the teaching of the sciences be included in this plan, the problem becomes a more serious one. The shaping of the curriculum is not the only problem; it will be necessary to train teachers for this kind of work. How many teachers of mathematics in the high schools of the United States to-day are qualified to do this? The work has become so highly specialized that in many schools there are teachers of geometry

[1] Such a plan is recommended by M. Laisant in *La mathématique*, p. 227. See Chapter IV of this work for a discussion of the teaching of geometry.

who teach only that subject. We must uproot this narrow specialization; let the high-school teacher of geometry teach arithmetic, algebra, and trigonometry as well. A better alternative would be for the teacher of mathematics to teach science also, and so secure a fitting correlation between these related branches. Teachers must be trained for this work, and, as Professor Moore suggests, the colleges should undertake such training. Under such a plan as the above, there may be a reduction of subject-matter in the several branches taught. If this be true, so much the better. The individual subjects may not be rounded out so symmetrically as formerly, but what is better will be the development of a well-balanced mind that is alert to the true significance of things.

Concerning the sequence of teaching the mathematical subjects it should be noted that:

a. We have copied unwisely from history, sometimes neglecting the good and accepting the bad.

b. The present order in the United States is not the historic one. It is more nearly so in France and in Germany.

c. We should follow the historic development and begin geometry before algebra.

d. The essentials of algebra should be understood by the time proportion is reached in geometry.

e. Trigonometry should be begun before plane geometry is finished, preferably when proportion is being applied

f. Thus there is opportunity for close correlation between the several mathematical branches when it can be the most effective.

g. By such a plan of correlation, the student sees a purpose in his work.

h. A closer correlation between the teaching of mathematics and science is desirable.

4. *The enrichment of the work in elementary mathematics by the introduction of phases of the higher work.*—Professor Perry has suggested that much of geometry may be assumed as axiomatic so as to give more time to higher mathematics, where the students will get real mental food. Professor Klein of Göttingen advocates the enriching of the subject-matter of secondary mathematics. The same requests are being made in Italy, in France, and in our own country. To what extent is this feasible or desirable? It is clear that such a procedure is impossible as

our school work is now organized. Where mathematics is required of all students, there are sure to be many for whom such advanced work would be meaningless. With less required mathematics in the high school, those who elect it in the higher courses would be in a position to use analytics and the calculus. Where mathematics is taught with reference to its applications in science, as Professor Perry would have it, there would be opportunity for the practical applications of certain phases of advanced mathematics. This seems wise provided it is used as a tool, and students are not required to master its logical associations. Such an addition to the subject-matter of high school mathematics adds nothing to the field of synthetic geometry *per se.*

5. *The enrichment of the subject-matter of geometry by the introduction of modern geometry.*—In recent years the text-books on geometry have added some phases of modern synthetic geometry. Regarding the development of modern geometry, Professor David Eugene Smith says:

"The nineteenth century has seen a notable increase of interest in the geometry of the circle and the straight edge, a geometry which can, however, hardly be called elementary in the ordinary sense. France has been the leader in this phase of the subject, with England and Germany following. Carrying out the suggestion made by Desargues in the seventeenth century, Chasles, about the middle of the nineteenth century, developed the theory of anharmonic ratio, making this the basis of what may be designated modern geometry. Brocard, Lemoine, and Neuberg have been largely instrumental in creating a geometry of the circle and the triangle, with special reference to certain interesting angles and points. How much of all this will find its way into elementary text-books of the next generation, replacing, as it might safely do, some of the work which we now give, it is impossible to say. The teacher who wishes to become familiar with the elements of this modern advance could hardly do better than read Casey's Sequel to Euclid (London, fifth edition, 1888). . . ."

"Among the improvements which affect the teaching of elementary geometry to-day, a few deserve brief mention. Among these is the contribution of Möbius on the opposite senses of lines, angles, surfaces, and solids; the principle of duality as

given by Gergonne and Poncelet; the contributions of De Morgan to the logic of the subject; the theory of transversals as worked out by Monge, Brianchon, Servois, Carnot, Chasles, and others; the theory of the radical axis, a property discovered by the Arabs, but introduced as a definite concept by Gaultier (1813) and used by Steiner under the name of 'line of equal power'; the researches of Gauss concerning inscriptible polygons, adding the 17- and 257-gon to the list below the 1000-gon; . . . and the researches of Muir on stellar polygons. . . ."[1]

Our best texts to-day have included some little of the subject-matter of modern synthetic geometry, but usually in the form of isolated propositions. The theorems of Menelaus, Carnot, and Pascal are of sufficient importance to be brought into the elementary work. We also usually find some exercises on the radical axis, and the "nine-point circle" is assigned as an original. A glance at Casey's "A Sequel to Euclid" will show what a large number of theorems and exercises we teach to-day which are not found in Euclid.[2] All this of course is not classed as "Modern Geometry." To what extent any systematized treatment of modern geometry should enter the school work, no definite answer can yet be given.

Two assumptions in Euclidean geometry mark points of departure for other systems of geometry. One of these is the postulate that two intersecting straight lines cannot inclose a space.[3] If this be denied it is possible to work out a logical geometry of the plane which has some characteristics in common with the geometry of the sphere. Such a system has been called elliptic geometry.[4] The parallel axiom,[5] which is equivalent to assuming that two intersecting straight lines cannot both be

[1] Smith, *The Teaching of Elementary Mathematics*, pp. 231-232. For a more complete treatment, see Smith, *History of Modern Mathematics.*

[2] For a history of the "nine-point circle," see Mackay, *History of the Nine-point Circle*, Edinb. M. S. Proc., XI, pp. 19-57. For the history of other familiar problems and theorems not given by Euclid, see articles by Professor Mackay in Edinb. M. S. Proc., V, VIII, IX, and XX.

[3] Euclid himself did not employ this, but tacitly assumed it.

[4] Also called Riemannian geometry on account of its great development by Riemann.

[5] The equivalent of this appears as Euclid's 5th postulate. It is sometimes called his 11th (or 12th) axiom.

parallel to a given straight line, marks another point of departure from our ordinary geometry. The denial of this axiom leads to the non-Euclidean geometry of Lobachevski and Bolyai (cir. 1825). Saccheri (1733), however, was the first to suggest investigating the subject in a scientific way.[1]

Should non-Euclidean geometry find a place in the elementary teaching? The teacher should be familiar with the subject for the sake of its bearing on Euclidean geometry and for the interest which it might arouse in the regular work, but young minds are not interested in the foundations of mathematics.

6. *The nature of the text-book.*—Texts of elementary geometry which were essentially academic began to appear in the eighteenth century. The nineteenth century has produced a great number of these, while the last ten years has seen an enormous production of texts, the character of which shows that there is at hand a general awakening to the need of better teaching.

Up to the last century, the geometries still copied Euclid in one respect. All of the propositions were explained, thus leaving little chance for the originality of the pupil. In recent times, the originals ("riders" of the English) have occupied a prominent place in the books. Recent geometries place these, to a great extent, as exercises among the proved propositions, but we still have a few texts that put them at the ends of the different books. Some texts even have placed as exercises, to be worked out by the students, propositions fully explained by Euclid. Why should not the simple theorem, "If two lines are parallel to a third line, they are parallel to each other," be left to the student for proof? On the other hand the student might well have "hints" on the more difficult originals.

Euclid first set us the example of giving demonstrations in full. According to De Morgan, we have gone beyond the "Elements" in this respect. In the sixteenth century the idea was current that Euclid gave only the enunciations and that his commentators later supplied the proofs. Many of the books of that century copied what they thought was the real plan of

[1] Smith, *History of Modern Mathematics*, p. 566. Also Loria, *Della varia fortuna di Euclide*. For a more complete historical account, see Stäckel und Engel, *Die Theorie der Parallelinien von Euclid bis auf Gauss*. The works of Lobachevski and Bolyai have been made accessible in English by Professor George B. Halsted. See his *Lobachevski's Non-Euclidean Geometry* and *Bolyai's Science Absolute of Space*.

Euclid. The French texts perhaps come nearer to the "Elements" in their easy, essay style. The opposite to this essay style is seen in the form of demonstration adopted in some of our American texts, where the proof is outlined under various "steps." Some texts have adopted a "developmental" method whereby the student supplies the answers to a series of questions. As Professor Smith points out, "the printed questions usually admit of but a single answer each, and hence they merely disguise the usual formal proof."[1] Provision should be made, however, for developmental work. Frequently a suggestion may be made regarding the nature of the proof, the student completing it. There should be completed proofs also to serve as models, especially as phases of new work enter.

The modern text-book should put the teacher and pupil not only in touch with the best methods, but should bring into relief other improvements that affect the teaching to-day. This will necessitate some attention to certain important theorems not found in Euclid and a recognition of such conceptions as the one-to-one correspondence between algebra and geometry, the law of converse, and reciprocal theorems. Regarding the elements which should be included in a text-book on geometry, Professor Smith says there should be "(1) A sequence of propositions which is not only logical, but psychological; not merely one which will work theoretically, but one in which the arrangement is adapted to the mind of the pupil; (2) Exactness of statement, avoiding slip-shod expressions as, 'A circle is a polygon of an infinite number of sides,' 'Similar figures are those with proportional sides and equal angles,' without other explanation; (3) Proofs given in a form which shall be a model of excellence for the pupil to pattern after; (4) Abundant exercises from the beginning, with practical suggestions as to methods of attacking them; (5) Propædeutic work in the form of questions or exercises, inserted long enough before the propositions concerned to demand thought—that is, not immediately preceding the author's proof."[2] In a small way only can the text-book bring about fundamental reforms in the teaching of geometry. It can simplify the logic, it can bring out the significant relations between the parts of the subject-matter, it can introduce new

[1] Smith, *The Teaching of Elementary Mathematics*, p. 255.

[2] *Ibid.*, p. 255.

material, but it is the trained teacher only who can bring the materials of the science into vital connection with every-day experiences. At the most the text is a thing of logic. Under the touch of the teacher it may either remain a mere machine or become a living reality to the student.[1]

7. *The question of method.*—The methods of attack used in elementary geometry to-day follow the lines laid down by the Greeks, who invented and used the methods of analysis, reductio ad absurdum, and exhaustions. To them is also due the conception of geometric locus, which was used by them rather sparingly in the solution of problems. Modern practice puts more emphasis on the employment of this effective method. It is of particular value in that it is a method of discovery. Many theorems in geometry are proved when the student knows full well that they are true. That such work is necessary is not to be denied, but ample provision should be made for the solution of geometric problems, in particular where the intersection of loci can be employed. Time should not be put on the proving of the necessary theorems on loci to the exclusion of their applications. Above all the discussion should not be omitted in such work. Here the student becomes an investigator in the best sense of the word. For example, in the problem to draw a circle with a given radius tangent to a given circle and a given straight line, the discussion, based on the intersection of loci, is exceedingly rich in material for this kind of work.

We shall now consider the term method in its broader significance. Ever since the time of Socrates, the method of question and answer has found service in the school room. Such a method gives opportunity for a developmental process. At the same time it may be utilized in an extreme dogmatic sense. In the early secondary schools this dogmatism found favor, where rules were valued more than logic.

It is perhaps safe to say that more attention is given to-day to the demonstrative teaching of geometry than ever before. When geometry was first taught, it was in the higher classes of the different institutions. As has been shown, it has been gradually taught to younger students, and pretty much in the

[1] For mention of worthy present-day texts in geometry, both for the elementary school and the secondary school, see the chapters on geometry in Smith's *The Teaching of Elementary Mathematics*.

same old way. The methods have not changed sufficiently to meet the new conditions. It is right that the logical aspect of the subject should not be neglected. We teach geometry largely as a training for the mind, but in this we must remember a psychological question is involved. Young boys and girls should not study geometry as was done by mature minds centuries ago. But that is essentially what we are asking them to do when we try to adapt students' minds to the logical demands of geometry, instead of adapting the subject-matter of geometry to the minds of the students, which is the demand of psychology. The proper pedagogical procedure is to secure the right balance between the psychological and the logical. Develop the mind logically but consistently with its needs and demands. Some minds take to the logical naturally, and hence a strict demonstrative system of teaching affords the proper training. But abstract reasoning is generally enjoyed only by the few. The majority of pupils are more interested in practice, particularly if this has a motor phase, and this interest should be appealed to. Pure logical work is heartily disliked by many students, especially girls. As Professor Dewey suggests, there may be more "ultimate logical worth" in merely *doing* things than in *thinking* along lines that are repugnant.[1]

The efforts being made by a few teachers to make the teaching of geometry experimental[2] is a move in the right direction. Such a system should be able to preserve the logical value of the subject and, at the same time, recognize the psychological principles involved. We have teachers who have employed the "heuristic" method, in which the pupil works out everything for himself. It has an inductive character and in that sense is experimental. Such a method has value, but particularly so when the proper problems arise which act as incentives to stimulate logical inquiry. This is now practiced somewhat in the upper grades of our elementary schools, but it is almost unknown in our high schools. Even without the aid of physics and astronomy, the principle can be carried out. The beginning work in the high school should be independent of any text-book. The

[1] Dewey, *The Psychological and the Logical in Teaching Geometry*, Educational Review, 1903, pp. 389-399.

[2] Used here synonymously with inductive work, including the use of some forms of instruments.

teacher and pupils should develop the work independently of any such restriction. The use of the scale, protractor, and triangle is as necessary as the Euclidean rule and compasses. The work can be made experimental in two ways, either in an indirect way by testing the truth of suggested theorems and seeking to discover further relations, or better, by associating the theory with some field problems. Every school which can afford it should have a transit for surveying purposes; or better still, a plane-table made by the students will give admirable service. The field problem can be easily chosen to include some proposition in the class work. The habit of performing these experiments after the theory has been learned is not the best plan. The experiments should precede the theory, so as to give an incentive for logical study. Even after the logical work is well advanced, this experimental work should not be abandoned. In this field work proper notes should be taken and careful drawings made to scale.

We recall that Archimedes proved propositions in two ways, by pure geometry and by experimental methods. He compared, for example, the areas of two plane figures under the supposition that they had equal weights. Such a method is entirely in accordance with good pedagogy to-day. The use of squared paper is to be recommended in checking up the theorems in mensuration. All work that has thus far been called experimental becomes real laboratory[1] work, as the term is used in science, where individual and not class work predominates.

The class hour in the United States is generally used for the "hearing" of lessons. In Germany and France, on the contrary, the work is developed in the class-room under the guidance of the teacher. Logical training is the aim and it is rigidly adhered to. In Germany especially, the text-book is consulted after the lessons have been developed. With us, lessons are assigned in advance. In Germany, the class as a unit takes part in demonstrating a theorem. With us, the individual receives the attention. Good teachers have used this principle of individualism to advantage by allowing certain interested students to carry on special investigations. A proper balance between

[1] Professor Safford in 1888 recommended the teaching of mathematics by the laboratory method. See Safford, *Mathematical Teaching and its Modern Methods.*

the extreme individual and class methods should produce the best results.

With respect to method we may conclude that:

a. We need the various methods of attack given us by the Greeks. The use of intersection of loci and of analysis is of especial value.

b. The method of question and answer is of value when not used dogmatically.

c. The inductive method should precede and accompany the demonstrative.

d. Practical experimental work gives rise to proper incentives for logical investigation.

e. The first work in geometry should be independent of any text.

f. The class hour should be a time for investigation rather than for the "hearing" of lessons.

g. The best features of the individual and class methods should be maintained.

CONCLUSION

The development of the geometric consciousness of the race is duplicated to a marked degree in the mental growth of the child. Recognizing that the child should study practical and intuitive geometry before the logical, we should distribute the work in geometry throughout the school years to conform to this development. Hence the work in the first school years should be a study of geometric forms. This should be comparative in its nature and out of it should be developed the idea of measurement. The purposefulness of measurement is brought out in mensuration when the pupil has had sufficient work in arithmetic. Experimental geometry finds a fitting place in the mensuration of geometric figures, at which time the pupil becomes familiar with the various instruments of construction. The way is thus paved for an inductive geometry, which is the fitting propædeutic for the deductive study in the secondary school.

In a general way, the race has passed through similar stages to reach the plane of geometric logic. We may say that the student has reached the Grecian stage when he begins the study

of deductive geometry. Notwithstanding that Euclid was not generally adhered to in succeeding centuries, and notwithstanding that geometry was to a certain extent taught with reference to its applications, much of our secondary teaching to-day makes the pupil live entirely in the Grecian age. There is need of a wider recognition of the unity of the mathematical subjects, in particular with reference to their common applications.

The teacher who desires to reform the teaching of geometry should remember:

1. That the mathematical subjects should not be taught in isolation from one another.

2. That modern synthetic geometry should have some claim for recognition.

3. That the race preceded the study of logical geometry by the practical.

4. That the history of the teaching of geometry shows that in succeeding centuries logical geometry has been taught gradually to younger students and that modern practice does not sufficiently recognize this fact.

5. That the psychological as well as the logical aspect of the teaching of geometry should be considered.

6. That inductive geometry should precede and to a certain extent accompany the deductive study.

7. That the teaching of geometry should have an experimental character.

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